# History of Nonlinear Oscillations Theory in France (1880-1940)

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*to Elisa…*

"If we wish to foresee the future of mathematics, our proper course,
is to study the history and present condition of the science. [1]. »

1. Henri Poincaré, *Science and Method*, 1914, p. 25.

# Introduction

From the end of the 19th century until the middle of the 1920s, the term "sustained oscillations" designated oscillations that are produced by systems moved by an external power such as maintained pendulum. It also referred to oscillations that are produced by self-sustaining systems such as the series-dynamo machine, the singing arc, or the triode. The numerous researches conducted in the domain of oscillations in France and around the world during this time period have never been the subject of an in-depth study. Until now the historiography has primarily been focused on Balthazar Van der Pol's contribution entitled : "On relaxation-oscillations" (Van der Pol 1926d). In this publication he introduced this terminology in order to distinguish a specific type of sustained oscillation and the history of relaxation oscillation appears to establish itself with his work. In his essay titled *Mathématisation du Réel* (*Mathematisation of Reality*), Giorgio Israel announces as follows :

> "Whilst searching for an explanation of how a triode assembled as an oscillator functioned, Van der Pol realized that the standard mathematical equations for oscillations were unusable" (Israel 1996, 39)

Similarly, David Aubin and Amy Dahan-Dalmedico offer the same considerations in an article titled "Writing the History of Dynamical Systems and Chaos" :

> "By simplifying the equation for the amplitude of an oscillating current driven by a triode, van der Pol has exhibited an example of a dissipative equation without forcing, which exhibited sustained spontaneous oscillations :
>
> $$\nu'' - \varepsilon(1-\nu^2)\nu' + \nu = 0$$
>
> In 1926, when he started to investigate its behavior for large values of $\varepsilon$ (where in fact the original technical problem required it to be smaller than 1), van der Pol disclosed the theory of relaxation oscillation."(Aubin and Dahan 2002, 289)

In her article "Le difficile héritage de Henri Poincaré en systèmes dynamiques" ("The difficult legacy of Henri Poincaré regarding dynamic systems"), Amy Dahan-Dalmedico goes further stating that "Van der Pol had used Poincaré's concept of limit cycles [5]" (Dahan 2004a, 279). Indeed, based on

5. See also Dahan (2004b, 237).

a historiographical reconstruction Van der Pol's contribution (1926d) seems to have been on several levels. Firstly the discovery of the relaxation oscillations produced by a triode, secondly, the equation of the phenomenon, and lastly the theoretical formalization linked to Poincaré's work (1881-1886) most significantly : the "limit cycles theory [6]". However, this historical representation does not accurately reflect reality and provides yet another illustration of the "Matthew effect [7]" : by focusing almost exclusively on Van der Pol and a few of his publications [8] dealing with the *free oscillations of a triode*, this historical reconstruction resulted in the partial, or even complete overshadowing of previous studies about sustained oscillations, as well as an overestimation of Van der Pol's contribution regarding the *forced oscillations of a triode* [9]. Moreover, the crystallization occurring around Van der Pol's article (1926d) has caused a misunderstanding about his own results. The term "relaxation oscillation" was indeed not introduced and defined in 1926, but rather in an article published in Dutch the previous year (Van der Pol 1925, 793). The analysis of Van der Pol's works (1927b, 1927c, 1930) shows that the *relaxation oscillation* phenomenon was first observed in 1880 by Gérard-Lescuyer in a *series-dynamo machine*, then in 1905 by Blondel in the *singing arc*. It will later be shown that the equation modeling for the triode oscillations was not carried out by Van der Pol in 1926, or even in 1922, as Giorgio Israel suggests :

> "Of particular importance in our case is the 1922 publication in collaboration with Appleton, as it contained an embryonic form of the equation of the triode oscillator, now referred to as "van der Pol's equation"." (Israel 2004, 4)

Van der Pol's article "A theory of the amplitude of free and forced triode vibrations", (Van der Pol 1920, 701), originally published in Dutch in 1920, points out that :

> "
> $$\ddot{x}-(\alpha-3\gamma x^2)\dot{x}+\omega x = 0 \tag{4}$$
> This equation has been previously considered * in connexion with the subject of triode oscillations.
>
> * Van der Pol, *Tijdsch . v. h. Ned. Radio Gen.* i. (1920) ; *Radio Review*, i. page 701 (1920). Appleton and Van der Pol, Phil. Mag. Xliii. page 177 (1922). Robb, Phil. Mag. Xliii. page 206 (1922)." (Van der Pol 1926d, 979)

In fact, Blondel (1919b) found the equation (4) symbolizing the free oscillations of a triode one year earlier, under a different form.

Lastly, the idea that Van der Pol developed relaxation oscillation theory, which originated in Van der Pol's (1922, 1926 ; 1934) and Philipe Le Corbeiller's articles (1931a, 1932, 1933) appears ground-

6. See also Poincaré (1882, 261).
7. See Robert Merton (1968).
8. More specifically, "On relaxation-oscillations", (Van der Pol, 1926d).
9. This question has however been briefly addressed by Pechenkin (2002, 272) and Israel (2004, 9), who also uncovered Van der Pol's and Van der Mark's work (1928a, 1928b) regarding the mathematical description of heartbeats. See Israel (1996, 34) and Israel (2004, 14). See Part III.

less. Indeed, Van der Pol did not use the concept of *limit cycle* in 1926 and did not cite [10] Henri Poincaré's works (1881-1886), as noted by Mary Lucy Cartwright (1960, 371). Van der Pol mentions it only after the publication of a note from Andronov to the C.R.A.S. [11] titled "Poincaré's limit cycles and the self-oscillation theory [12]" (Andronov, 1929a) -in which he suggests that the periodic solution of a self-oscillator matches one of Poincarés' *limit cycles*- during a series of lectures at the École supérieure d'Électricité, on the 10th and 11th of March 1930.

> "On each of these three figures, we can see a closed integral curve, which is an example of what Poincaré called a limit cycle ( [13]), because the neighboring integral curves approach it asymptotically." (Van der Pol 1930, 16)

Thus, in spite of an undeniable wish to develop a mathematical theory on the relaxation oscillation as early as 1926, Van der Pol did not manage to set its founding principles. By establishing a link between Poincaré's work (1881-1886, 1892) and this type of oscillation, Andronov (1929a) incorporated it into a broader viewpoint : self-oscillation theory, or self-sustained oscillation theory [14].

Consequently, it becomes clear that Van der Pol's contribution (1926d) did not consist in the modeling of the equations for the free oscillations of a triode, which would have let him develop a relaxation oscillation theory, but rather in the discovery of a new type of oscillation, and naming it *relaxation oscillation*. Van der Pol's crucial role (1926d) therefore resides, on the one hand, in the conceptualization [15] of an oscillatory phenomenon possessing two distinctive time scales when given specific parameter values, or in other words, two types of evolution : one slow, one fast. On the other hand, he describes a great number of seemingly differing processes using a single differential equation in an undimensionalized form. The duality of the "slow-fast" phenomenon became part of self-oscillation [16] theory, which Andronov (1928, 1929a, 1929b, 1930a, 1930b 1935, 1937) developed based on Poincaré's work (1881-1886, 1892). Therefore, amongst others, Andronov's note (1929a) is seen as a remarkable event in the history of nonlinear oscillation theory by scientists [17] as well as historians of science [18].

> "And it is only in 1929 that Russian researcher A.A. Andronov suggested that self-

10. In his article, Van der Pol (1926d, 981) names this closed curve "periodic solution".
11. *Comptes Rendus de l'Académie des Sciences de Paris*, or Proceedings of the Academy of Sciences.
12. Contrary to Dahan's (2004a, 279 ; 2004b, 237) and Diner's (1992, 339) affirmations, this two-pages note absolutely cannot be his "graduation work", i.e. his "senior thesis". See infra Part II, 27.
13. Andronov (1929a).
14. In his original version, written in Russian, Andronov(1929b) did not use the phrase *self-sustained oscillations*, but rather the term *self-oscillation*, from the Greek *auto* and Russian *kolebania* : oscillations. See Pechenkin (2002, 288).
15. Part II will demonstrate that Van der Pol was at the root of the concept of relaxation oscillation.
16. Part II will establish the difference between *self-sustained oscillations* and *relaxation oscillations*.
17. See Mandelstam *et al.* (1935, 83) ; Abelé (1943, 18) ; Rytov (1957, 170).
18. See Diner (1992, 340) ; Dahan (1996, 21) ; Aubin (2002, 13) ; Pechenkin (2002, 274) ; Dahan (2004a, 279) ; Dahan (2004b, 237).

oscillations are expressed as limit cycles from Poincaré's theory. This date leads to a new era in this field of study." (Minorsky 1967, 2)

This milestone had not been questioned until now. Indeed, doing so would have meant finding a publication predating Andronov's where the same type of link between an equation similar to Van der Pol's - *i.e.* defining the self-oscillation phenomenon- and Poincaré's *limit cycles* was discussed. A study of most of the articles and books published during the period before Andronov's note has been carried out. The works of French researchers such as Alfred Liénard (1928, 1931), Henri and Élie Cartan (1925), Paul Janet (1919, 1925), and André Blondel (1919a, 1919b, 1919c, 1920, 1926) have proven to be of great interest by raising once again the question of Poincaré's scientific legacy. Whilst these works did not lead to the discovery of a link similar to the one Andronov had introduced, they showed that it was another device, older than the *triode*, that should have been the subject of research : the *singing arc*. Thus far, this device had played an essential part during the rise of the wireless telegraphy [19], approximately between 1900 and 1914. During this period, Poincaré published several crucial studies on *wireless telegraphy*. (Poincaré, 1902a, 1902b, 1907, 1908, 1909, 1911). At a series of lectures at the *École Supérieure des Postes et Télégraphes* in 1908, twenty years before Andronov (1928, 1929a, 1929b), Poincaré (1908, 1909) set the connection between his own work on the *limit cycles* and the differential equation symbolizing the *self-oscillations* occurring in the *singing arc*. Thus, from the end of the nineteenth century to the middle of the twentieth century, three [20] devices - the *series dynamo machine*, the *singing arc*, and the *triode* - were the seat of a new type of oscillatory phenomenon, then considered as sustained oscillations, before Van der Pol (1925, 1926d) named it *relaxation oscillation* and before Andronov (1929a, 1929b, 1930a, 1930b 1935, 1937) incorporated it in the *self-oscillation theory*. During this period, these three devices became the subject of much research in France and all over the world, the aim being :

– to isolate the cause of this phenomenon,
– to schematize the device's *current-voltage characteristic* [21] and
– to model the equation symbolizing this new type of oscillation in order to determine its amplitude and period.

19. *Télégraphie Sans Fil*, or *T.S.F.* in France.

20. Around the same time, Léauté (1885) also noticed this type of oscillation in a hydraulic sifting control device. See Appendix 2.3.

21. Thompson (1893, 247), stated that the term *characteristic* was introduced by Marcel Deprez (1843-1918) in 1881. See Deprez (1881, 893). The *current-voltage characteristic* of an electric dipole is the function relating the tension at its terminals with the intensity of the current flowing through it. With direct current, the characteristic of a resistance ($R$) is a linear function of the intensity : $u = f(i) = Ri$. The inclination of this straight line provides the value of the resistance, seen in this case as a constant. The characteristic of an electromotive force (*e.m.f.*) generator, and an internal resistance (r) is a linear function of the intensity : $u = f(i) = E - ri$. The $y$-intercept represents the *e.m.f.* ($E$), also assumed to be constant. These relations are the mathematical expression of Georg Ohm's law (1789-1854).

The schematization of the *current-voltage characteristic*, *i.e.* the *e.m.f.* of each of these three devices, and the determination of the oscillation period were be the two main obstacles to overcome in order to define the relaxation oscillation phenomenon.

# Première partie

# From sustained oscillations to relaxation oscillations

# Chapitre 1

# From the series-dynamo machine to the singing arc : Gérard-Lescuyer, Blondel, Poincaré

## 1.1 The series dynamo machine : *the expression of nonlinearity*

At the end of the nineteenth century, *magneto-* or *dynamo-electric* machines were used in order to turn mechanical work into electrical work and vice versa. With the former type of machine, the magnetic field is induced by a permanent magnet, whereas the latter uses an electromagnet. These machines produced either *alternating* or *direct current* indifferently. They were therefore the most economical of all appliances where powerful currents are required, such as supplying lighthouses with power using electrical arcs [1]. A *dynamo-electric machine* where the electromagnet is integrated into the circuit is called a series *dynamo machine*, or *self-exciting dynamo* [2].

### 1.1.1 Jean-Marie-Anatole Gérard-Lescuyer's *paradoxical experiment*

There are few biographical elements regarding the man who conducted this experiment. His family name, Gérard-Lescuyer, probably originated from his father Jean-Baptiste Gérard marrying a woman called Marie-Anne Lescuyer in Paris. Jean-Marie-Anatole Gérard-Lescuyer was an engineer [3] and the director of a public liability electric company established in Courbevoie. From 1895 to 1902, he was a member of the French society of physics. Similarly, baron Paul Arnould Edmond Thénard (1819-

1. For more details see Hospitalier's (1884, 68-115) or Lemoine's work (1890, 19).
2. See Hospitalier (1881, 86) and Appendix 2.1.
3. On 10 September 1879, he invented an electric arc lamp and an automatic light, as shown in an article published in *La Nature* (Hospitalier 1881, 220-222) and signed E. H. who is in fact Édouard Hospitalier (1853-1907), engineer at the *Arts et Manufactures* and chief editor for the periodic publication l'*Électricien*. He also invented a new incandescent light bulb in 1885 (see Rodet 1907, 67) and a chainless bicycle (see Picture n° 1, 7)

1884), physicist and chemist, presented on July 26, 1880 his first and only scientific contribution : a note in the *Comptes rendus de l'Académie des Sciences de Paris* (*Proceedings of the Academy of Sciences*). This is where he detailed an experiment, describing it as an "electrodynamical paradox". However as time passed he lost its paternity.

Gérard-Lescuyer's research on electrical generators led to his invention of a machine named after him [4]. In all likelihood it also brought about his experiment associating a *dynamo-electric machine* used as a generator with a *magneto-electric machine*, which in this case can be considered as the motor. He reports on the found effects in the following way :

> "[. . . ] If the current produced by a dynamo-electrical machine is sent into a magneto-electrical machine, a strange phenomenon is witnessed. As soon as the circuit is closed the magneto-electrical machine begins to move ; it tends to take a regulated velocity in accordance with the intensity of the current by which it is excited ; but suddenly it slackens its speed, stops, and starts again in the opposite direction, to stop again and rotate in the same direction as before. In short, it receives a regular reciprocating motion, which lasts as long as the current that produces it." (Gérard-Lescuyer1880a, 226, 1880b, 215)

He also observed the periodical reversal of the magneto-electric machine's circular motion, despite the direct current, and wondered about the causes of this oscillatory phenomenon. He estimated that the change in the motor rotation can only happen if the current running through it also changes direction. He then researched how this inversion occurred :

> "Some extraneous cause, then, must arise to reverse the polarities of the inductors of the generating dynamo-electrical machine, so that this machine may immediately give rise to a current of an opposite direction, which reverses the rotation direction of the receiving machine." (Gérard-Lescuyer 1880a, 226, 1880b, 215)

He indeed notices this polarity reversal by placing small compasses near the inductors, and notes that "movements of the compass-needle coincide with those of the galvanometer". This enables the establishment of a cause and effect relationship between the polarity reversal and the appearance of a reverted current. Thus, in order to try and provide an explanation, he hypothesizes that "the receiving magnetoelectrical machine can, for some unknown reason, receive periodically an increasing of its velocity". The experiment's trial confirms this :

> "[. . . ] if our hypothesis proves right, this phenomenon will no longer occur when, by any means whatsoever, we prevent the receiving magnetoelectrical machine from increasing its velocity : applying a brake suffices to do so. However, as soon as the brake intervenes the preceding effects disappear." (Lescuyer 1880a, 227 ; 1880b, 215)

4. In his book, Hospitalier (1884, 106-107) described A. Gérard's machine. Patent n° 336 636, 23 February 1886 (U.S. patent). See also Boulanger (1885, 111 and 120).

According to him this increase in the velocity of the magneto-electric machine, by inducing a reverse electrical current, caused the inductors' polarity reversal and inverted the rotation. It was actually proven by du Moncel (1880) a few weeks later, then by Witz (1889a, 1889b), and by Janet (1893), that the gap situated between the brushes of the dynamo is the source of an electromotive force (*e.m.f.*), *i.e.* a potential difference at its terminals symbolized by a nonlinear function of the intensity that flows through there (see above note 18, 4). Therefore this "cause not investigated" by Gérard-Lescuyer, the essence of his paradox, is the presence of an *e.m.f*, which has a *nonlinear current-voltage characteristic* leading to *sustained oscillations* [5]. This reservation he emits is also notable :

> "What are we to conclude from this ? Nothing, except that we are confronted with a scientific paradox, the explanation which will come, but that does not cease to be interesting."
> (Gérard-Lescuyer 1880a, 227, 1880b, 215)

Gérard-Lescuyer's article was then published in the *Philosophical Magazine* with the title "On an electrodynamical paradox" (Gérard-Lescuyer 1880b), which allowed him to get some response in the United States particularly thanks to a paragraph in the *New York Times* of 22 August 1880 :

> "Gérard-Lescuyer finds that when the current from a dynamo-electric machine is sent into a magnetic electric machine the latter moves with increasing speed, then it slackens, stops, and turns in the opposite direction, and so on. The polarity of the inductors is reversed."

The New York Times which was founded in 1851 created very early on a column called *Scientific Gossip*, which aimed at retelling the most notable scientific events at the time. Later on this column would feature the announcement for the first *Congrès International d'Électricité* in Paris on the $11^{th}$ of October 1882 (*New York Times* of 10 September 1882), as well as an article titled "Music in Electric Arcs", about the discovery of the *singing arc* (see infra) by William Du Bois Duddell (*New York Times* of 28 April 1901). As for Gérard-Lescuyer's note to the *C.R.A.S.*, it seems its publication in the *Philosophical Magazine* on one hand and the use of the word "paradox" on the other are the reason it caught this reporter's attention. Aside from this type of journalistic echo, the "paradox" uncovered by Gérard-Lescuyer did not seem to garner immediate reaction from the physicists and engineers researching electrical phenomena [6]. However the oscillatory phenomenon noted and detailed for the first time here later caused curiosity due to its paradoxical nature.

### 1.1.2 Théodose du Moncel's *electrokinetic interpretation of the paradox*

Being interested in science and more specifically in electricity Viscount Théodose du Moncel (1821-1884) became an Electrician-Engineer for the *Administration des lignes télégraphiques fran-*

5. It was later established that they were actually *relaxation oscillations*. See infra, 78-79
6. Except for Théodose du Moncel.

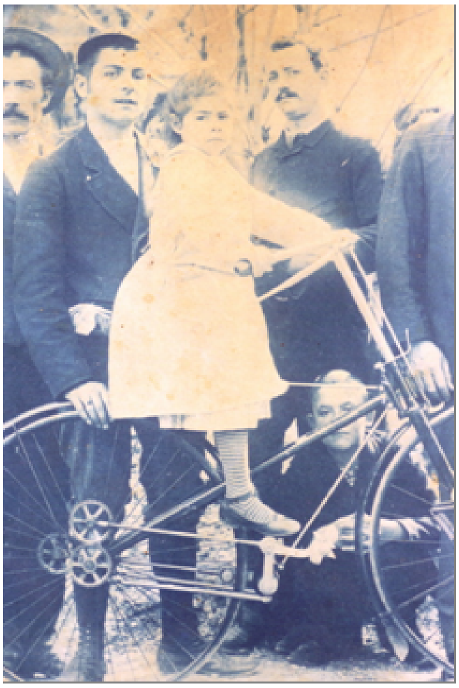

FIGURE 1.1 – J.M.A. Gérard-Lescuyer (left) and his daughter Marguerite in 1884.
Document uploaded online by his great-grandson.

*çaises* (French telegraph lines administration) during the second half [7] of the nineteenth century. He submitted several notes to the *French Academy of Science* and became a member on the 21$^{st}$ of December 1874. He wrote numerous books [8] and articles, the latter published in the periodicals l'*Électricité* and *La Lumière Électrique*. He published on the 1$^{st}$ of September 1880 an analysis of Gérard-Lescuyer's experiment titled "Réactions réciproques des machines dynamo-électriques et magnéto-électriques" ("Reciprocal reactions of dynamo-electric and magneto-electric machines") (Moncel, 1880). He then used the concept of counter-electromotive force in order to explain the inductors' polarity reversal found by Gérard-Lescuyer :

> "The author of these experiments attributes this effect to a periodical increase in the magneto-electric machine's speed, which would generate a counter-electromotive force greater than the electromotive force developed by the dynamo-electric machine, able to reverse the inductor's polarity. A new reversed current would therefore be produced by the dynamo-electric machine, which would then turn off the current produced by the counter-electromotive force and as a consequence cause it to stop. This would be followed by movement in the opposite direction and so on. In order to verify this explanation,

7. In his youth, he went to visit the Greek archeological sites, and sold his sketches to earn money. See the *Annales Archéologiques*, 1848, volume eight, page 179 and page 236, *Bureau des Annales Archéologique*, Paris, *Librairie Archéologique de Victor Didron*. See Cornelius Herz's obituary (1884).
8. See Moncel (1858, 1872-1878, 1878, 1879, 1882).

he forced the magneto-electric machine to move steadily by using a brake. In these conditions the machine's rotation became regular." (Moncel 1880, 352)

Gérard-Lescuyer's description enabled du Moncel to prove the existence of an electromotive force (*e.m.f.*) developed by the dynamo and of a counter-electromotive force (*c.e.m.f.*) generated by the periodical increase in speed which reverses the inductors' polarity and the motor's rotation. He then explained that this inversion happens as soon as the magneto's *e.m.f.* is superior to the dynamo's *e.m.f.* He did not use the scientific paradox concept but rather "reciprocal reactions" as indicated by the article's title. As soon as September 1880 his analysis pushed Gérard-Lescuyer's experiment towards rationality by involving electrical quantities, which would later allow Witz then Janet to provide a thorough explanation of the phenomenon.

FIGURE 1.2 – Count Th. du Moncel, from Herz (1884, 383.

### 1.1.3 Aimé Witz's *geometrical interpretation of the paradox*

Because of their difference in status Gérard-Lescuyer and Aimé Witz (1848-1926) had entirely different approaches : whilst the first focused on practical and technological applications, the second aimed for a purely theoretical approach. Witz was both a Doctor of Science and an engineer from the *Arts and Manufactures* school and taught at the *Faculté libre des sciences* in Lille, where he later became dean emeritus [9]. He was an expert in thermodynamics and electricity. In 1889 and 1890 he published a series of articles in which he went back over Gérard-Lescuyer's experiment and offered a heuristic explanation solving the paradox. The first article was a note published in the *C.R.A.S.* on the 6$^{th}$ of July 1889 titled "Polarity reversals in series-dynamo machines" (Witz 1889a). This article explained the principles of geometrical construction explaining the observed phenomenon but the corresponding graphical representation was unfortunately absent due to the format constraints of the *C.R.A.S.* Witz developed his method in a longer and more detailed article published in December 1889 in the *Journal de Physique théorique et appliquée* : "Des inversions de polarité dans les machines série-dynamos" ("Research on series-dynamo machines polarity reversals"), (Witz 1889b). This article provides interesting insight from two different angles. The first one is historical since Witz explained how he came to learn about this experiment :

> "This perplexing phenomenon which I had thought unknown had already been observed by Mr. Gérard-Lescuyer, as was pointed out to me by Mr. Hospitalier. With this phenomenon, this eminent physicist had found a paradox, which he gave up on explaining in 1880. Today this fact appears less mysterious, as we will attempt to demonstrate after examining it more thoroughly." (Witz 1889b, 582)

Surprisingly no reference is made to Théodose du Moncel's work either by Witz or Hospitalier [10] despite the latter's apparent familiarity with them. The second angle is theoretical since this article took a second step towards the rational explanation of the phenomenon by using a geometrical construction. Witz's approach was thus based on experiments whereas Gérard-Lescuyer's (1880a) was purely phenomenological. For instance he began his study with the following words :

> "Firstly this experiment can be reproduced in a laboratory with any series-dynamo activating a machine with a separate exciter or a magneto-electric machine."
>
> (Witz 1889b, 582)

Thus, Witz shows that this experiment was not unique but actually perfectly reproducible, that is to say that its carrying out did not require specific conditions. Following scientific methods Witz measured the voltage and intensities and then analyzed the evolution of these quantities to change the approach from qualitative to quantitative.

***Principle of Witz's construction***

With this construction, Witz (1889b) provided a geometrical version of du Moncel's analytical explanation (1880). Indeed, he drew the current-voltage characteristic of the dynamo, *i.e.* the curve

9. For Witz's biography, see for example Charles Lallemand's obituary (1926).
10. See Hospitalier (1884, 64, 98, 196, 220, 224, 237, 255). It should be noted that between the first and third edition of his work, Hospitalier greatly reduced his references to du Moncel.

representing the variations in voltage at its terminals according to the intensity of the current running through it [11]. Although it appears on Figure I.1, which is a simplified version of Witz's original construction, that this curve possesses the characteristics of a cubic plane curve (two extrema and one inflexion point), the next step, *i.e.* its modeling *via* a mathematical function would require a lengthy process, as it will be demonstrated later. He then drew the characteristic of the magneto [12], which can be compared to a motor. It is a straight line and its y-intercept depends on the motor rotation speed. This line shifts in parallel with itself according to the values of the motor rotation speed. Thus the two characteristics : the dynamo's (curve) and the motor's (straight line) show a number of intersections. The amount and value of these intersections depend on the straight line's placement in relation to the curve.

> "[...] this line intersects the characteristic curve of the generator at points C and D. The abscissa for C indicates the actual intensity in the circuit. This intensity will stay constant as long as the machine's speed is kept the same. But when decreasing the resisting torque of the motor, its speed will instantly increase, the characteristic ordinates go up, and the counter-electromotive force increases. p'q' replaces pq, since points C and D become closer and merge. The secant line has become tangent, then intersects only with the symmetrical branch. The current intensity in the circuit has gone down at the same time, gradually decreasing to zero and suddenly dropping to a negative value."
>
> (Witz 1889b, 585)

His construction shows that the polarity reversal occurs as soon as the motor's *c.e.m.f.* becomes greater than the dynamo's *e.m.f.*, which causes the motor's oscillations. Indeed, the cubic plane curve allows the intensity to take negative values, which would not be possible if it was a straight line. In the circuit the series-dynamo (*i.e.* a component comparable to a "negative resistance [13]") plays a role similar to a pendulum's friction, but its sign's alternating between positive and negative sustains the oscillations instead of damping them. Therefore, Witz's work in 1889 broke through a second threshold in the understanding of the phenomenon using a graphic representation highlighting the nonlinearity of the dynamo's *e.m.f.* However whilst his construction explains how this polarity reversal observed by Gérard-Lescuyer occurs, it did not provide the cause as he pointed out :

> "All in all, by explaining the diagram we can find all the defining features of the phenomenon [...]. It is not a paradox anymore. However, let us not delude ourselves : the previous reflections show how things happen, but not why they happen in this way"
>
> (Witz 1889b, 586)

11. This is the *e.m.f.* of the dynamo.
12. It is the *c.e.m.f.* generated by the increase in speed. See du Moncel (1880, 352).
13. The concept of "negative resistance" will be defined in the second paragraph.

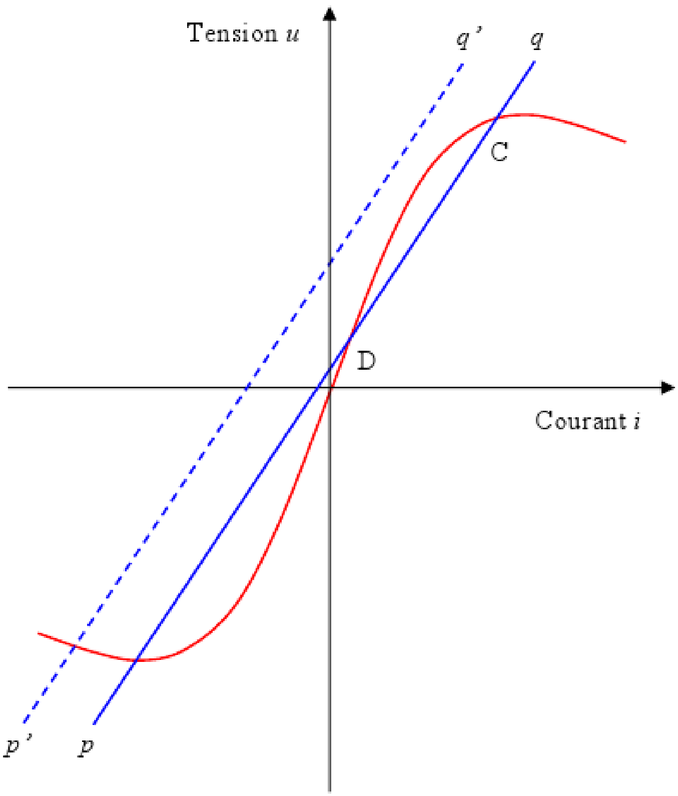


FIGURE 1.3 – Characteristics of the dynamo (red) and the motor (blue).

Nevertheless, it must be noted that Witz came very close to the explanation provided by Janet (1893) less than four years later :

> "Lastly, let us prolong the line tangent to the characteristic curve until it meets the symmetrical branch of this curve : the y-intercept of the intersection gives the value of the electromotive force developed right after the inversion of the rotation and the pole reversals." (Witz 1889a, 1245)

### 1.1.4 Paul Janet's *incomplete equation modeling (I)*

After graduating from the *Ecole Normale Supérieure* at age twenty-two and then passing the *agrégation*, Paul Janet (1863-1937), philosopher Paul Janet's son (1823-1899) defended his doctorate dissertation titled "Étude théorique et expérimentale sur l'aimantation transversale des conducteurs magnétiques" ("Theoretical and experimental study on the transverse magnetism of magnetic conductors") in front of the *faculté des Sciences de Paris* in 1890. He was then appointed as *maître de conférences* (lecturer) in Grenoble where he opened the very first course on Industrial Electricity against dean François Raoult's advice. It was met with such success that as soon as the next school year started François Raoult had to request for this course to be made official and for an electrical engineering laboratory to be created. Two years later Paul Janet became the head of the *École Supérieure d'Électricité* he had just founded and kept on teaching there. Therefore, as early as 1893, Gérard-Lescuyer's experiment [14] was already being quoted by Janet in his *Industrial Electricity* course as an important example then in 1900 in volume I of his *Leçons d'Électrotechnique Générale* :

> "If the current produced by a series generator is sent to the armature of a motor with a separate exciter, it will start up and there will be a certain amount of counter-electromotive force developing there. If enough speed is gathered this counter-electromotive force can become greater than the electromotive force of the generator, the current will be reversed as well as the generator's polarity. The motor's armature will then abruptly stop and start in the opposite direction and the same phenomenon occurs with notable periodicity. The frequency depends of course on the motor's excitation. It must be noted that in order to fully explain the phenomenon three electromotive forces must be involved during the variable period : the electromotive force of the generator, the counter-electromotive force of the motor, and the electromotive force of the inductor(1).
>
> ---
>
> (1) (1) We encourage the reader to try and put the problem into equation, which is fairly simple : find either the current's voltage, or the motor speed in relation to time. "(Janet 1900, 222)

He explained that in addition to the *c.e.m.f.* of the motor and the *e.m.f.* of the generator found by du Moncel (1880) and Witz (1889a, 1889b) the *e.m.f.* of the coil inductor must also be taken into account. In this way, as seems to be shown in the footnote(1) above, he had already set up the incomplete differential equation characterizing the phenomenon generated by Gérard-Lescuyer's experiment. However the issue concerning modeling the complete equation for this new type of oscillation, which required the modeling of the (*nonlinear*) *current-voltage characteristic*, was still unsolved. At this point in time, a second device was introduced.

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14. Janet did not cite Gérard-Lescuyer's (1880a, 1880b), nor du Moncel's (1880), nor Witz's works (1889a, 1889b). He did not mention Raoul Lemoine's book (1890) either. This book, titled L'Électricité dans l'industrie is reminiscent of his lecture given in 1893, in which the experiment was explained, but attributed to Witz. See Lemoine 1890, 21).

FIGURE 1.4 – Paul Janet in 1923, rue de Staël, from Jacques Boyer / Roger Viollet.

## 1.2 The singing arc : *sustained oscillations*

### 1.2.1 William Du Bois Duddell's *revision of Thomson's formula*

At the end of the nineteenth century a forerunner to the incandescent light bulb called *electric arc* [15] was used for lighthouses and street lights. Regardless of its weak glow it had a major drawback : the noise generated by the electrical discharge which inconvenienced the population. In London, physicist William Du Bois Duddell (1872-1917) was commissioned in 1899 by the British authorities to solve this problem. He thought up the association of an oscillating circuit made with an inductor L and a capacitor C (F on Fig. 1.5) with the *electrical arc* to stop the noise (see Fig. 1.5). Duddell (1900a, 1900b) created a device that he named *singing arc* [16]. He was able to determine that the frequency of the musical sound [17] emitted by the arc corresponds to the natural frequency of the associated oscillating circuit and it was expressed using Thomson's formula (1853) : $T = 2\pi\sqrt{LC}$ ($T = 2\pi\sqrt{FC}$ since $C = F$).

15. The electric arc (artificial, in opposition to lightning's thunderbolts) is associated to the electric discharge produced between the extremities of two electrodes (made of carbon for example), along which comes light emission. It is still used nowadays for theater projectors, thermic plasma, as well as in the metalworking industry for "arc soldering" or metal fusion (electrical arc furnace). See for example Vacquié (1995).
16. For a *brief history of the arc*, see Hertha Ayrton's book (1902, 19).
17. If its frequency is within human hearing range.

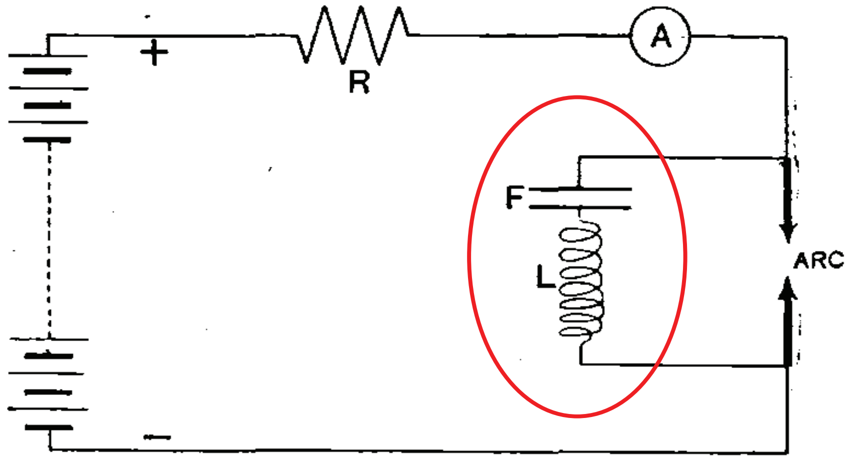


FIGURE 1.5 – Diagram of the *singing arc*'s circuit, from Duddell (1900a, 248)..

Duddell had actually created an oscillating circuit capable of producing not only sounds (hence its name) but especially electromagnetic waves. This device would therefore be used as an emitter for wireless telegraphy until the triode replaced it. The *singing arc* or *Duddell's arc* was indeed a "spark gap" device meaning that it produced sparks which generated the propagation of electromagnetic waves shown by Hertz's experiments [18] as pointed out by Poincaré :

> "If an electric arc is powered by direct current and if we put a self-inductor and a capacitor in a parallel circuit, the result is comparable to Hertz's exciter." (Poincaré 1907, 79)

### Conditions for starting the oscillations sustained by the singing arc

After his discovery Duddell kept on studying the *singing arc* aiming to generate electromagnetic waves able to send a signal. Duddell (1900a, 268 ; 1900b, 310) then showed that in order to establish the oscillation speed, two conditions must be met :

$$\frac{du}{di} < 0 \qquad (D_1)$$

$$\frac{du}{di} > r \qquad (D_2)$$

meaning that the oscillations occur if : the derivative for the difference in electric potential at the arc's terminals in relation to the current going through it is negative (condition $D_1$) and if this derivative is greater (in absolute value) than the internal resistance $r$ of the parallel circuit (see Fig. 1.5). This implies that if the *current-voltage characteristic i.e.* the arc's *e.m.f.* is nonlinear and has a

18. Carried out from 1886 to 1888 by German physicist Heinrich Rudolf Hertz (1857-1894), this experiment, or rather series of experiments proved the existence of electromagnetic waves predicted by James Clerk Maxwell (1831-1879) during the previous decade.

"falling" or decreasing part, oscillations may occur (see infra, 19). These two conditions have been subsequently analyzed throughout many studies. One of them was a note from Paul Janet (1902b) presented to the *Académie des Sciences*, in which he found Duddell's two conditions through other means concluding with another practical use for the singing arc :

> "Duddell's singing arc offers a remarkable way of using a constant electromotive force in order to obtain alternative current." (Janet 1902b, 823)

Duddell's conditions ($D_1$, $D_2$) were very quickly questioned by several researchers such as Maisel (1903, 1904, 1905) who believed they did not play any role in the phenomenon :

> "As a matter of fact this condition does not come into consideration at all in phenomena of this kind, as Maisel has shown by both theoretical and experimental researches."
> (Ruhmer 1908, 178)

The same went for engineer Aimé Williame (1906), who published a very detailed note in the *Annales de la Société scientifique de Bruxelles* titled "On the singing arc theory" in which he explains :

> "*Condition* $\dfrac{dv}{di} < 0$. – In short, despite both theory and experimentation showing that the consequences ensuing from the condition $dv/di < 0$ cannot be accepted without restrictions, until now nothing has shown that this condition was not necessary from beginning, for $i = 0$. However, our current knowledge, relating to the quotient $dv/di$, includes consequences that have not been proven to be exact in all cases." (Williame 1906, 188)

These authors also questioned the value of the oscillations' frequency which Duddell modified in 1901.

### Frequency of the oscillations sustained by the singing arc

Indeed in response to some comments from the readers of the periodical *The Electrician* Duddell (1901c) recalled that during his first experiments he had overlooked the circuit's resistance ($R$) and when taking it into account, the frequency of the sound emitted by the arc corresponding to the circuit's specific frequency given by Thomson's formula (1853) had to be modified following another formula which has since then been named after him (see Table 1.1).

The first studies carried out in France then seemed to validate these formulae. Thus for Charles Fabry (1903, 376) the classic formula (Thomson's) seemed to be "indeed verified by Mr. Tissot's experiments (1902)". Whereas Blondel (1905c), in a dissertation "On the singing arc phenomenon", explained by using other experiments that :

> "The singing arc's frequency is essentially variable and not properly defined ; while in the case of the continuous phenomenon it can be given approximately by Duddell's formula, in the case of the discontinuous phenomenon, it is however quite different from the eigenfrequency of the oscillating circuit." (Blondel 1905c, 102)

| Thomson's formula (1853) | Duddell's formula (1901c) |
|---|---|
| $T = 2\pi\sqrt{LC}$ | $T = \dfrac{2\pi}{\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}$ |

TABLE 1.1 – Duddell's and Thomson's formulae for the singing arc's frequency.

Meanwhile, Williame (1906) considered that the quotient $dv/di$ must vary either in a continuous or discontinuous manner and thus he concluded that :

> "In both cases it is not proven anymore and it even seems improbable that the current would be sinusoidal in the parallel circuit and that its frequency could be deduced using Thomson's formula." (Williame 1906, 187)

Two years later in an article entitled "The frequency of the singing arc", which starts with a very thorough bibliographical study of the various research carried out in this field, George Nasmyth (1908, 122), using new experiments demonstrated that the arc's frequency varies in relation to the inductance $L$ and also varies depending on the current flowing through it. Nasmyth (1908, 140) then offered a new formula for the frequency which took into account the derivative for the arc's *e.m.f.* However at this time the researchers had to face another obstacle : "measuring or even defining the resistance of an arc". (Fabry, 1903, 376)

## 1.2.2 Edlund and Luggin's work *on the concept de "negative resistance"*

Too many variables (diameter, constitution, and width between the carbons of the arc) and the phenomenon called *arc hysteresis* (see infra, 18) made it almost impossible to reproduce the experiments exactly. Therefore determining the arc's physical quantities such as its resistance and the voltage at its terminals was a real problem for the scientists at the end of the nineteenth century. However it was observed that the potential difference $u$ on the one hand increases somewhat quickly with the distance $l$ between the carbons and on the other hand it no longer obeys Ohm's law (see footnote 18, 5). Consequently an empirical relationship, *linear* then *nonlinear*, between the voltage $u$ at the arc terminals, the intensity $i$ running through it, and the distance $l$ between the carbons (see Table 1.2 *infra* [19]) is established. This potential difference expresses the arc's *e.m.f.* To explain the importance of this potential difference between the carbon electrodes, some researchers such as Edlund (1867) then later Duddell (1901a, 1901b, 1904) concluded that there must exist a *counter-electromotive force*

19. In Table 1.2, a, b, c and d represent four constants which depend on the material and diameter of the carbons.

in the arc. This led them to consider that the arc's resistance, already seen as a variable, could allow negative values. According to Ayrton (1902, 54) and Child (1909, 233) the concept of "negative resistance" was introduced by Hans Luggin (1863-1899) who studied the voltage at the arc terminals using the Wheastone bridge method, which at the time, was used for measuring the arc's resistance. A slight increase in the electromotive force applied to an ordinary conductor, through which a current flows, produces an increase in voltage at the terminals and results in a difference in electric potential, with a positive sign. Luggin (1888) however showed that on the contrary in the arc can be found an increase in electric current which is followed by a decrease in voltage between the carbons' extremities and consequently the potential difference generated by an increase in the electromotive force has to have a negative sign. From this he concluded :

"[...] der Lichtbogen habe also einen negativen Wiederstand [20]."

(Luggin 1888, 568)

| Author | Relation | Type |
|---|---|---|
| E. Edlund [1867] | $u = (a + bl)\, i$ | Linéaire |
| S. P. Thompson [1892] | $u = a + \dfrac{bl}{i}$ | Non linéaire |
| H. Ayrton [1895] | $u = a + bl + \dfrac{c + dl}{i}$ | Non linéaire |

TABLE 1.2 – Relation between the arc's voltage and intensity.

In an article published in 1909 in *Physical Review* Child tried to clear up the confusion surrounding this terminology :

> "There is no denying that the resistance of the arc is of a negative quantity but of all the uses of the word this is perhaps the least justifiable and to speak of a negative resistance is, to say the least, misleading. The word resistance means primarily something which hinders the movement of some object. An electrical resistance means something hinders the flow of an electric current and the most natural meaning of the expression "negative resistance" would be something which helps the flow of the current. It is needless to say that the resistance of the arc does not help the flow of the current." (Child 1909, 233)

20. "[...] hence, the arc possessed a negative resistance."

As for Mrs Ayrton (1902, 75-76) she recalled that during his meeting with the *British Association* in Ipswich in 1895, her husband William Edward Ayrton (1847-1908) submitted a study on the “Arc’s resistance” in which he got the same results as Luggin despite not knowing about them. Whilst Edward Ayrton did not publish these works [21] they were nonetheless faithfully transposed by Frith and Rodgers (1895) during a presentation in front of the *Physical Society* in May 1896. At the turn of the nineteenth century the electric arc was defined as having three characteristics : “negative resistance”, varying electromotive force, and counter-electromotive force. Blondel’s research on the subject confirmed the first two and disproves the third entirely.

### 1.2.3 André Blondel’s work and the *non-existence of a c.e.m.f.*

In France the electrical installations of the maritime signals and beacons were in disrepair. Seeing this André Blondel (1863-1938), a young engineer assigned to the *Service central des Phares et Balises Balises* (Central Office of Lighthouses and Shore Lights), started researching the electric arc to improve the structures. In order to do this he designed the galvanometric oscillograph [22] (see Appendix I.5). Blondel (1891, 1892, 1893c, 1894, 1897) dedicated the first part of his work to this oscillograph which helped "the electric arc theory to take a major step forward" (Bethenod 1938b, 751). In 1883 twenty-year-old Blondel joined the *Ecole Polytechnique* after having graduated from the *école normale supérieure* the same year. He then joined the *Ecole des Ponts et Chaussées* in 1885 and graduated first in his class in 1888. Following the acquisition of his licence in Mathematical Sciences in 1885 and in Physics in 1889 he attended Poincaré’s lectures, before starting his career as an engineer for the *Service central des Phares et Balises*. At the time, this position fell under the management of the *Direction Générale des Ponts et Chaussées*. This is most likely the reason why he turned to researching the electric arc, “in order to find a specific application of lighting” (Blondel 1891, 552) but also and mainly to shed light on the arc’s nature. Blondel (1891, 621) then introduced the concept of *oscillation characteristic* of the direct current arc (see infra) in order to study its stability. He conducted a series of studies on the direct and alternating current arcs and came to the conclusion that an *electromotive force* did exist in the direct current arc (depicted by one of the relations in Table 1.2), as well as resistance variability :

21. In 1893 during a stay in Chicago, William Edward Ayrton lost the only copy of an article representing three years of work as seems to be corroborated by the *Proceedings of the International Electrical Congress held in the city of Chicago*, 1893, vol. 1, 258 : “The manuscript was partially destroyed by fire through a most unfortunate accident.” From his wife, Hertha Ayrton (1854-1923) :

> “[. . . ] Prof. Ayrton’s ill-fated Chicago paper, which, after being read at the Electrical Congress in 1893, was accidentally burnt in the Secretary’s office, whilst awaiting publication.” (Ayrton 1902, vi).

Hertha Ayrton, who assisted him during his research, took over and published the results in the periodic *The Electrician*, (Ayrton, 1895), and in her book *The Electric Arc*, (Ayrton, 1902). See also Trotter (1924).

22. Since Blondel’s oscillograph (1893a, 1893b) enabled the study of alternating currents, it was modified and improved by Duddell (1897), then replaced by Braun’s cathode-ray oscilloscope (1897).

> "[. . . ] the shapes of the periodic curves for the current and the difference in electric potential at the terminals show that the arc's resistance varies in relation with the current [. . . ]" (Blondel 1897, 515)

However in 1893 he rejected (see Blondel ; 1893c, 614) the theory stating that the arc held a counter-electromotive force and in 1897 he managed to disprove it :

> "I believe these measures make it absolutely clear that the electric arc markedly behaves like a resistance and that it does not feature what usually defines a counter-electromotive force comparable to the observed potential difference ([2]). It is therefore not caused by and electrolysis phenomenon.
>
> ---
>
> ([2]) This does not mean this resistance is the same as an ordinary resistance, but the resulting effects are equivalent [. . . ] " (Blondel 1897, 519)

Blondel then specified in a footnote that in Ayrton's formula (1895), which he nonetheless considered to be the most accurate and adequate, that :

> "The term $a + \frac{c}{i}$ would be the one symbolizing the counter-electromotive force."
> (Blondel 1897, 515)

It therefore appeared that the electric arc's resistance was indeed variable or even negative and that the arc held an electromotive force represented as a nonlinear function determined empirically (see Table 1.2) but that it did not feature a counter-electromotive force.


FIGURE 1.6 – André Blondel. Stamp drawn and engraved by Jules Piel (14 septembre 1942).

## 1.3 The "arc hysteresis" phenomenon : *hysteresis cycles or limit cycles ?*

By observing a simplified diagram of Duddell's layout (1900a, 248) we can notice, as Janet did, (1902b) the superposition of two currents in the arc branch : one is constant and produced by the direct current generator, the other is alternating and produced by the parallel circuit ($LC$).

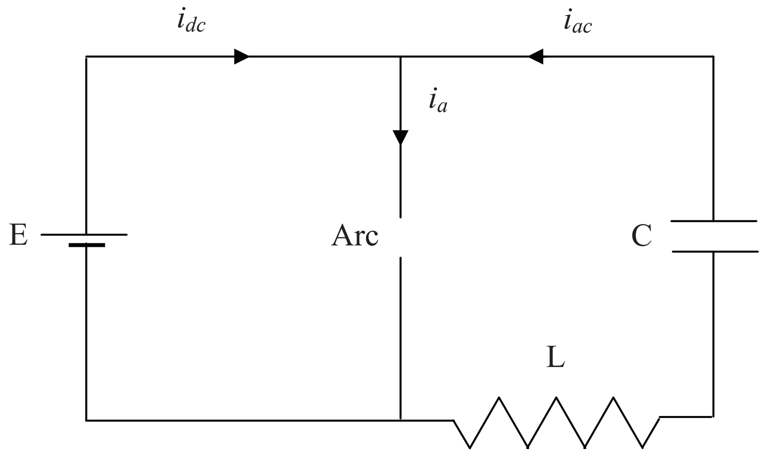


FIGURE 1.7 – Simplified diagram of Duddell's *singing arc* circuit.

### 1.3.1 The static and dynamic characteristics of the arc

The term characteristic was introduced by Marcel Deprez (1881, 893) (see *supra*, footnote 17, 4) to refer to the curve representing the function relating the voltage at the terminals of a dipole and the intensity of the current traversing it. Blondel (1891, 621) then associated the term with the word oscillation in order to define the stability of the direct current arc. One of the issues with the electric arc is that it can be powered by either direct or alternating current. Moreover, even with D.C., it is traversed by a current which is the result of the superposition of these two types. This led to a differentiation in the arc's characteristics.

#### Characteristic of the direct current arc

Therefore when the arc is powered by a direct current generator its *e.m.f.*, which can be represented by one of the relations in Table 1.2 is called *oscillation characteristic* or *static characteristic*. As for Thomson's formula (the most commonly used at the time) : $u = a + \frac{bl}{i}$ this characteristic is shaped as an equilateral hyperbola (in red on Fig. 1.8).

This representation was already problematic at the time since the use of low-intensity currents invalidated Thomson's formula and made it necessary to "connect" the hyperbola (in red on Fig. 1.8)

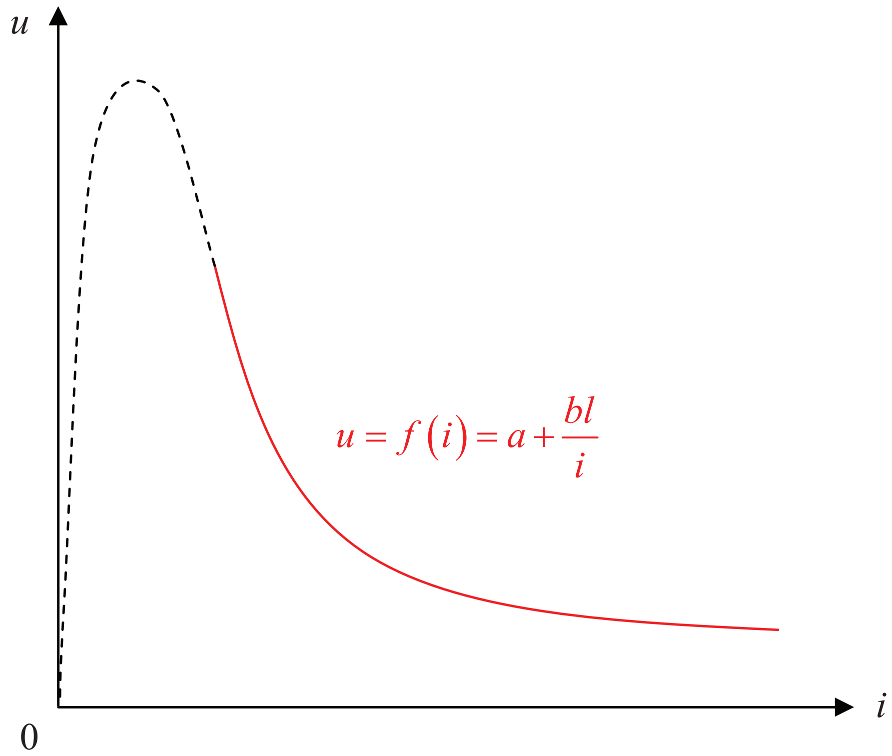


FIGURE 1.8 – Static characteristic of the *singing arc*.

to another function (dotted line on Fig. 1.8)) whose expression had not been yet determined. The characteristic was deemed as possessing a "falling" part, i.e. an increase in current corresponded to a decrease in voltage.

**Dynamic characteristic of the alternating current arc**

When an arc is powered by an alternating current its *e.m.f.*, which is not represented by any of the relations in Table 1.2, is called *dynamic characteristic*. In this case the potential difference in the arc, *i.e.* its *e.m.f.*, differs depending on whether the current flowing through it increases or decreases. This phenomenon, which Théodore Simon (1906) called arc *hysteresis*, was discovered by Mrs. Ayrton.

## 1.3.2 Hertha Ayrton's works

At the start of the twentieth century a major roadblock in Hertha Ayrton's research was the impossibility of recreating exactly the experiments on the arc.

> "When these experiments were first started, at the beginning of 1890, it was not known what were the conditions necessary for the P.D. [23] between the carbons to remain constant

23. Potential Difference.

when the current and length of arc were both kept constant, and consequently it was found, as had been found by all previous experimenters, that a given current could be sent through an arc of given length by many different potential differences, and that no set of experiments made one day could be repeated the next." (Ayrton 1902, 100)

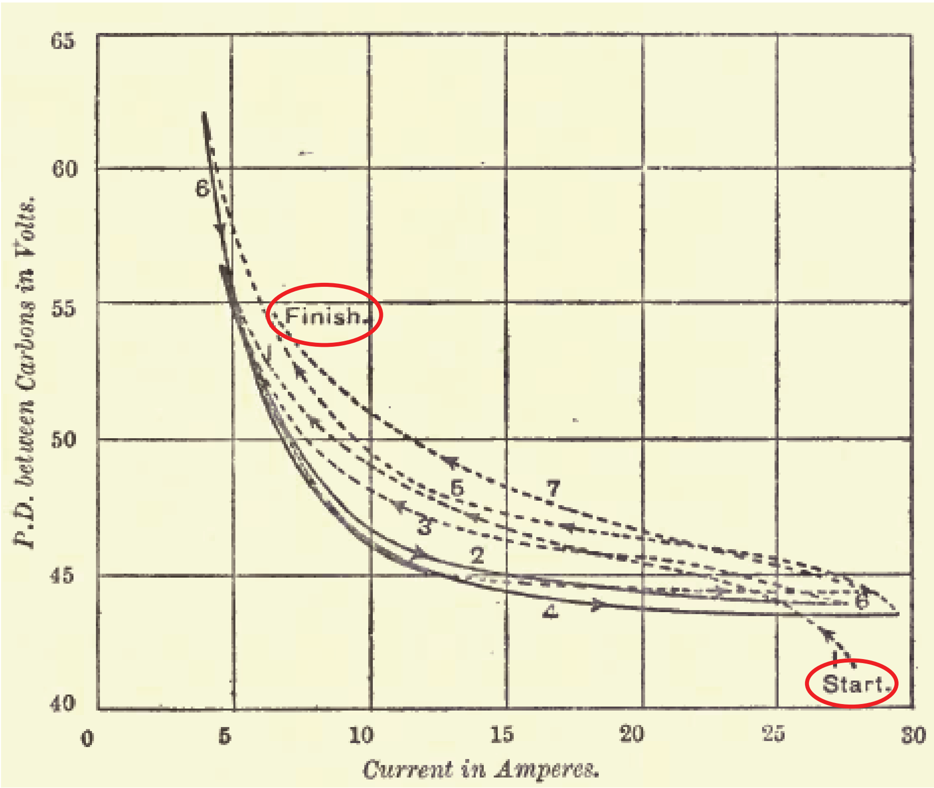


FIGURE 1.9 – Dynamic characteristic of the arc, from Ayrton (1902, 101).

With a slight increasing then decreasing of the intensity Mrs Ayrton noticed that the potential difference at the carbon terminals of the arc, i.e. its *e.m.f.* was greater when the current increased than when it decreased and that it did not vary in a straight line but rather in a closed curve which is represented on Figure I.4 following a dotted line starting at point 1 (Start circled in red on Fig. 1.9) till the finish point. Mrs Ayrton inferred that :

> "Hence, from these curves it would be impossible to find any exact relation between P.D. and current for a given length of arc." (Ayrton 1902, 101)

Mrs Ayrton's experiments on the alternating current arc on the one hand brought to light, based on the cyclical aspect of the *dynamic characteristic*, the hysteresis phenomenon, and on the other showed that it is impossible to establish a relation of the same type as the ones in Table 1.2 between the voltage at the arc terminals (its *e.m.f.*) and the intensity traversing it.

### 1.3.3 André Blondel's work *on the singing arc phenomenon*

During the following years Blondel (1905a, 1905b, 1905c) intended to "completely shed light on the phenomenon of the singing arc" (Bethenod 1938b, 752). Therefore almost ten years after his last publication titled "On the electrical arc phenomenon" (Blondel, 1897) he submitted the results of his research in a short note to the *C.R.A.S.* on the $13^{th}$ of June 1905 titled "Sur les phénomènes de l'arc électrique" ("On the singing arc phenomenon"). He then made a presentation on the $7^{th}$ of July 1905 in front of the *Société française de Physique*. It was better developed but would only be published in February 1906 in the *Journal de Physique Théorique et Appliquée*. In the meantime a detailed and unabridged version of his work was published in the periodical *L'Éclairage Électrique* in July 1905. By modifying the settings of Duddell's singing arc and using a bifilar oscillograph he designed, Blondel (1905a, 1905b, 1905c) found two new opposite types of singing arcs : one continuous, the other discontinuous.

– The first type (Fig. 1.10) "to which corresponds a quite pure sustained sound, and that is strictly speaking, Duddell's musical arc, gives rise to current curves in the arc and capacitor with continuous shapes, almost sinusoidal, without the intensity in the arc decreases to zero or at least remains zero during a considerable amount of time ; changes in voltage across the arc are contained within very close limits." (Blondel 1905b, 79).

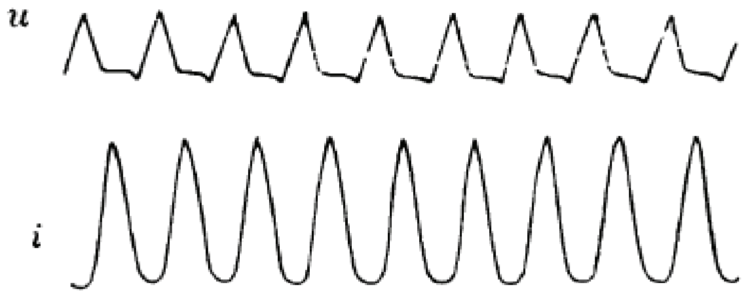


FIGURE 1.10 – Oscillographic record : first type curve : musical arc, voltage $u$ and intensity $i$ in the arc, from Blondel (1905b, 78).

– The second type (Fig. 1.11) "to which corresponds a shrill or hissing sound is a discontinuous phenomenon characterized by the fact that the arc current $i$ has angular points and appreciable zeros of time during which the current of charge $j$ presents ordinarily some flat spots, whilst the voltage between the electrodes $u$ undergoes a double oscillation of large amplitude the values of which go often below zero or above the electromotive force of the generator." (Blondel 1905c, 80).

Analyzing these records shows that the first type corresponds to sinusoidal oscillations, as also noted by Blondel, whereas the two time scales appearing on Figure 1.11 ("double oscillation") are typical concerning oscillations, which Van der Pol (1925, 1926a, 1926b, 1926c, 1926d) called a few

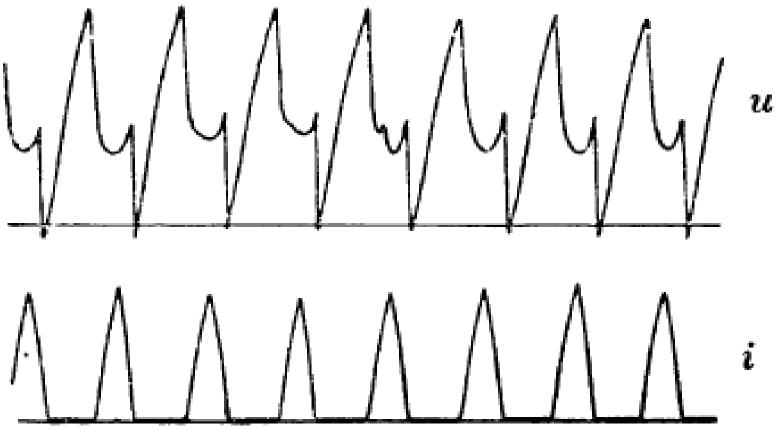


FIGURE 1.11 – Oscillographic record : second type curve : whistling arc, voltage $u$ and intensity $i$ in the arc, from Blondel (1905b, 79).

years later *relaxation oscillations* (see *infra*, 66). Moreover the way in which Blondel (1905b, 80) went from discontinuous to continuous regime by simply moving the carbons farther apart can be assimilated to Van der Pol's method (1926a, 1926b, 1926c, 1926d) which enabled him to switch from relaxation oscillations to sinusoidal oscillations by modifying the parameter $\varepsilon$ (see *infra*, 67).

Blondel's circuit (1905a, 1905b, 1905c) was perfectly identical to Duddell's (1900a, 248), which has been simplified on Figure 1.7 (see *supra*, 18).

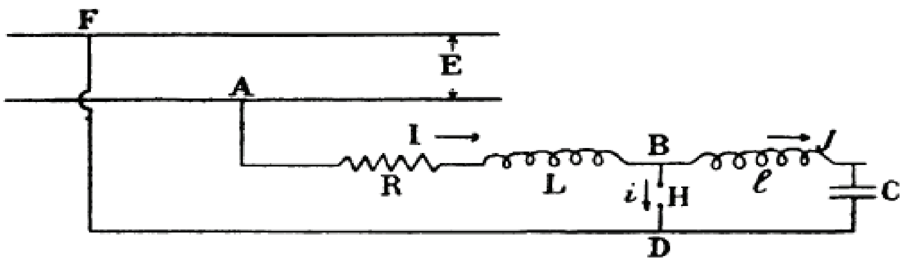


FIGURE 1.12 – Circuit diagram : $H$, the arc ; $C$, the capacitor ; $R$, the rheostat ; $L$ and $l$, self-induction ; ABDF, the power supply circuit produced by line power ; BCD, the oscillation circuit, from Blondel (1905b, 77).

Despite the D.C. power supply the current in the arc results from the superimposition of both alternating and direct currents. This is similar to regarding it as being traversed by direct current modulated by a sinusoidal fluctuation. One must therefore consider including these two characteristics. Blondel hence used on one hand the *static characteristic of the direct current arc*, which he named oscillation characteristic, in order to define the oscillation phenomenon he encountered with these two

types of sine waves (BcA curve, symbolized with a red dotted line on Figure 1.10). On the other hand, the *dynamic characteristic of the alternating current arc* since hysteresis also occurred, each value of the direct current in the arc being modulated by alternating current. An analogy can be made between the conjunction of these two characteristics and Ptolemy's depiction of epicycles.
As a matter of fact the potential difference at the arc terminals, *i.e.* its *e.m.f.*, obeys a hyperbolic law like Thomson's but the current modulation involves a cyclical evolution between two limit values for each point of this curve.

> "These phenomena can be easily explained by the arc's properties between homogenous carbons, in regards to the stability. Let BcA be the theoretical stability curve of an arc (potential difference variation law) when the current is decreased by increasing the power-supply resistance beyond the normal value (corresponding to the power-supply straight line $DM_1$)). As a consequence of the well-known phenomenon (see Ms. H. Ayrton, *The Electric Arc*) of the continuous-speed delay caused by the heating and cooling of the electrodes, when one modifies the voltage between two limit points $I_1'$ and $I\,\check{}_1$, the continuous-speed point $M_1$ is not exactly a short line but rather a small cycle anbqa."
>
> (Blondel 1905a, 1682)

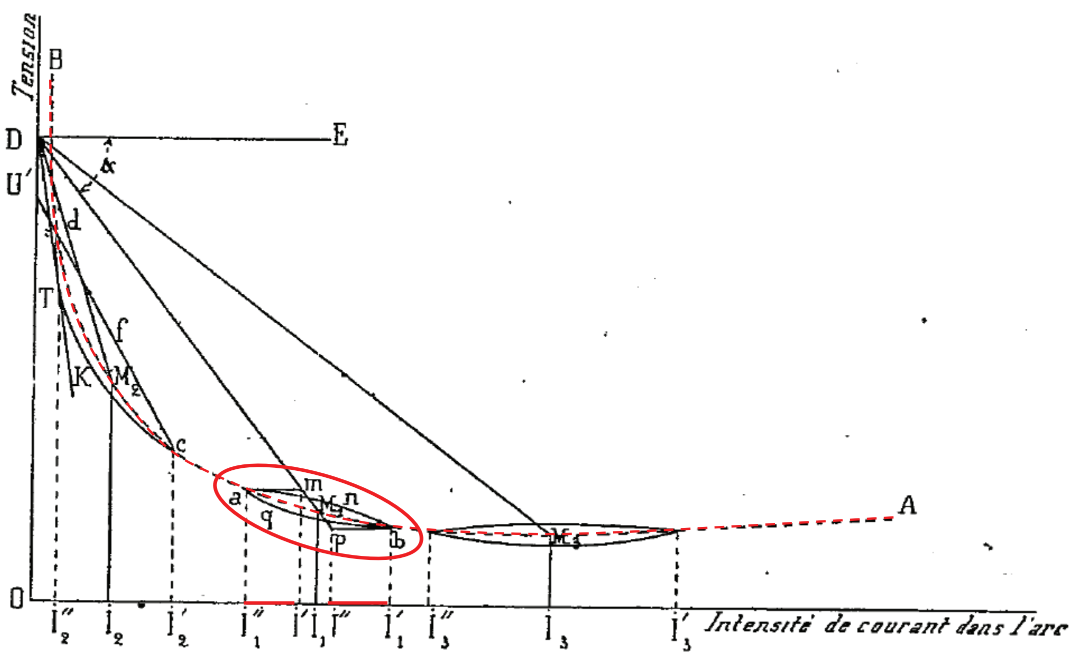


FIGURE 1.13 – Static and dynamic characteristics of the arc by Blondel (1905a, 1681).

These *hysteresis cycles*[24] were a first experimental depictions of the *limit cycle* concept in the phase plane $(i, u)$, which would allow Poincaré (1908) to provide evidence of the existence of oscillations sustained by the singing arc (see *infra*, 25) a few years later. However they could not be qualified as such since due to the impossibility of exactly reproducing the experiments their claims could not be verified. Using this graphic representation Blondel provided a condition for the oscillations and an explanation for the oscillatory phenomenon, for each noted type of sine wave :

> "For the oscillation to be possible the current exchanged between the arc and the capacitor must therefore make up for the difference between $\overline{I_1''I_1'}$ and $\overline{I'I''}$. The experiment for the first type, the musical arc, corresponds exactly to this case. The power-supply current can even still undergo weaker oscillations. When there is little induction in the power-supply circuit, part of the power-supply current serves to compensate the energy losses due to the law of heat loss or other causes in the oscillating circuit, due to the fact that the cycle charging branch $anb$ is above the discharging branch $bqa$. The capacitor thus receives more energy than it can return." (Blondel 1905a, 1682)

This description of a system (the arc) in which part of the produced energy was used to compensate for the losses and thus sustained the oscillations, was the early stage of the definition Blondel (1919d, 120) later provided for a *self-oscillating system*[25]. Lastly he concluded in regards to the oscillation frequency :

> "The frequency of the singing arc is essentially variable and ill-defined. With the continuous phenomenon it can be approximately determined using Duddell's formula but that is not the case with the discontinuous phenomenon where the formula has no relation whatsoever with the oscillating circuit's specific frequency." (Blondel 1905b, 54)

This indeed corresponds to what was noted in the case of the *relaxation oscillations* (see *infra*, 64).

### 1.3.4 Théodore Simon's work : *the hysteresis cycle*

In an article published in the periodic *Physikalische Zeitschrift*, Simon explains :

> "For a conductor to produce stable oscillations in a parallel circuit, its characteristic curve $e = f(i)$ must be falling which means the fall in potential decreases as the current increases [. . . ]. If the electromotive force varies between two limits, the characteristic[26] takes the shape of a closed loop, in some ways similar to hysteresis loops."
>
> (Simon 1906, 435)

24. It should be noted that Blondel (1905a, 1905b, 1905c) could not use this name, since this terminology was apparently introduced by Simon (1906) the following year.
25. It was then established that this terminology had been introduced by Blondel (1919d). See *infra*, 56.
26. This term refers to the dynamic characteristic. See *infra*, 19.

Simon corroborated the notion that the oscillation characteristic, *i.e.* the arc's *e.m.f.* must have a "falling" or decreasing section in order for the stable sustained oscillations to occur. He then carried out the calculations inherent to Blondel's description of hysteresis cycles (1905b, 54) to establish conditions for the oscillation.

### 1.3.5 Heinrich Barkhausen's work

The following year (1907) Barkhausen published a fundamental work on the problem of producing oscillations, especially electric oscillations [27]. He (1907, 46) also provided a graphic representation of the *dynamic characteristic of the alternating current arc* in the phase plane ($i$, $e$). He thus got a slightly more accurate *hysteresis cycle* than Blondel's (1905a, 1905b, 1905c). The frame at the center of 1.14 corresponds to the coordinate points $(i_0, e_0)$ where it is met by the *static characteristic* which was not drawn by Barkhausen (and was added in red). He then divided the plane in four sections where he defined a network of equilateral hyperbola $e_1 i_1 = const.$ which enabled him to find the hysteresis cycle and to deduce that if the characteristic is in quadrants II and IV some energy is added to the alternating current whereas the characteristic being in the two other quadrants (I and III) means that energy is taken from the alternating current. This result should be compared to the one Blondel established (1905a, 1682) (see *supra*, 23).

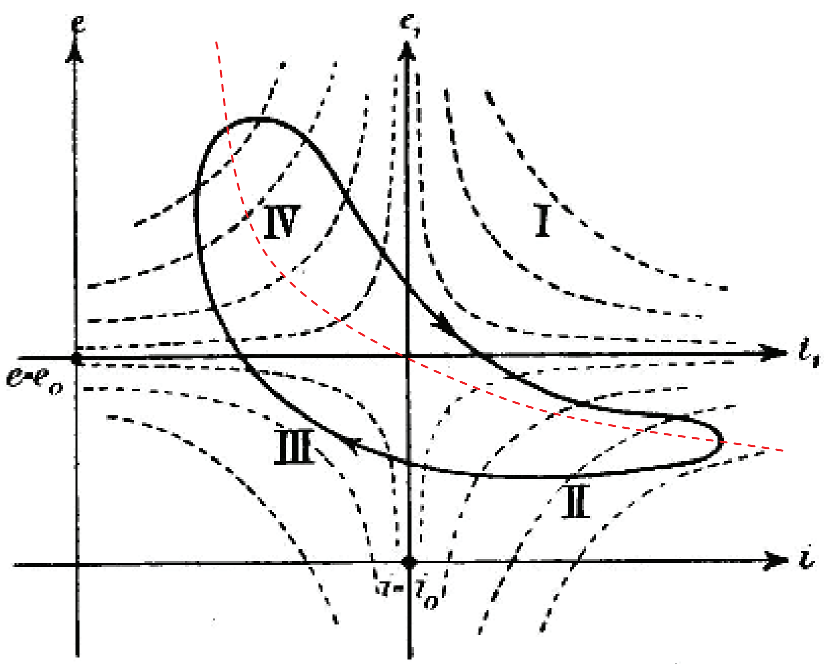


FIGURE 1.14 – Dynamic characteristic of the arc, from Barkhausen (1907, 46).

Barkhausen's work (1907) played a crucial role later on in the development of the theory of non-

27. Mechanical oscillations are only discussed in the last fifteen pages of this hundred-and-twelve-pages long book.

linear oscillations by the Russian school of thought, especially in the choosing of the terms used to refer to the new type of self-sustaining oscillations, *i.e. selbst schwingungen* (see *infra* Part II).

### 1.3.6 Ernst Ruhmer's work

In his work titled "Wireless Telephony" Ruhmer (1908, 148) also displays a diagram denoting the arc's *dynamic characteristic*

> "We thus find that on taking oscillographic records of the current and voltage of the arc, we obtain a dynamical characteristic which shows high voltages with increasing currents, and low ones with decreasing currents, forming a hysteresis loop." (Ruhmer 1908, 148)

He then describes the evolution of the arc's characteristic on the different parts of the hysteresis cycle and explains :

> "Whenever the capacity is charged it begins to discharge itself through the arc, thus increasing the latter until the maximum current is reached and the cycle recommences (A, Fig. 1.15)." (Ruhmer 1908, 149)

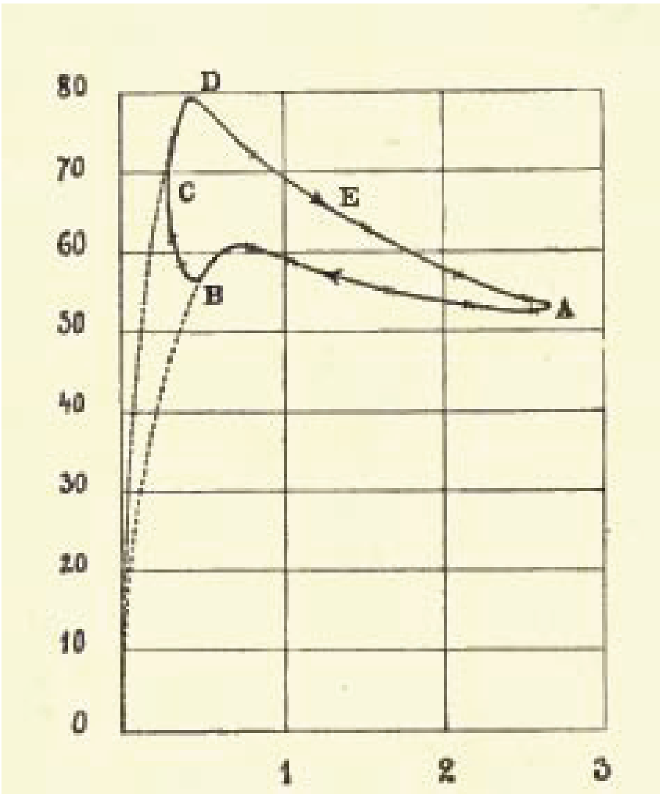


FIGURE 1.15 – Dynamic characteristic of the arc, from Ruhmer (1908, 148).

The relation between these hysteresis cycles and the existence of sustained oscillations, *i.e.* oscillations with limit cycles, therefore seemed definitely established (see *infra*, 31).

The characteristic properties of the arc are hence comparable to those of Gérard Lescuyer's *series-dynamo machine* and the problems to be solved were noticeably similar as a result : isolating the cause of the phenomenon, establishing a model for the arc's *current-voltage characteristic*, and putting these oscillations into equations in order to deduce their amplitude and period. Whilst the existence of an electromotive force in the arc and Ayrton's or Thomson's law of nonlinear variation seemed to provide solutions for these problems in regards to putting the arc oscillations into equations. Henri Poincaré is actually the one who carried it out in 1908 in a little-known study on wireless telegraphy.

## 1.4 Henri Poincaré's "forgotten" lectures *the limit cycles in 1908*

Henri Poincaré (1854-1912) already stood out in 1893 by providing a solution to the telegrapher's equation and was soon very involved in the wireless telegraphy development. In 1908 this mathematician and physicist had already published a huge number of articles on the subject (Poincaré, 1902a, 1902b, 1907) as well as a book entitled "La théorie de Maxwell et les oscillations hertziennes. La Télégraphie sans fil [28]" ("Maxwell's theory and Hertzian oscillations. Wireless telegraphy") published in 1904 then translated in English and German. It was according to Blondel (1912, 100) the "first truly scientific presentation" on the subject. As in any field he researched, Poincaré was a unanimously recognized expert, as evidenced by his relations with the two French wireless telegraphy specialists Gustave Ferrié and Camille Tissot [29], as well as his presence in numerous scientific committees such as the one for the periodic *La Lumière Électrique*. It is therefore not incidental that he was named president of the Development Council of the *École Supérieure des Postes et Télégraphes* (today Sup'Telecom Paris Tech) in 1901. The head of the school Édouard Estaunié (1862-1942) even used this in order to restore the school's prestige and it went far beyond all his expectations [30]. Poincaré's lectures addressed a broad array of subjects :

- Propagation de courants variables sur une ligne munie d'un récepteur (Propagation of varying currents on a line fitted with a receiver),

28. The first edition in 1899 did not include wireless telegraphy, and was simply titled "Maxwell's theory and Hertzian oscillations". Another edition was published in English in 1904 with the title "Maxwell's theory and Wireless Telegraphy", and was published in French in 1907.

29. We can mention that Poincaré was the examiner for the dissertation of Camille Tissot (1868-1917), defended in 1905, as well as his usual interlocutor on these subjects.

30. From Atten *et al.* (1999, 50) : "While the semester Henri Poincaré dedicates, every two years, to especially difficult subjects, brings in a large audience, it is probably not of particular interest to engineers of the Postes et Télégraphes. But this globally renowned mathematician-physicist gives a new prestige to the school, and it is E. Estaunié's aim : "When...I had to reorganize it (the school), it seemed to me that resorting to Poincaré would give me every chance to achieve my goals...he agreed to give a lecture for free on...a question related to electricity of our choosing and never addressed beforeĚ I must say that the simple announcement of his collaboration brought in numerous outsiders, showing the incredible reputation of the master and the appeal of such a program." Estaunié used his relations in order to invite other renowned scientists such as Pierre Curie (1859-1906).

– Théorie mathématique de l'appareil téléphonique (Mathematical theory of the telephone),
– La T.S.F. et la diffraction des ondes le long de la courbure de la Terre (Wireless telegraphy and the wave diffraction along the curvature of the Earth),
– La T.S.F. et la méthode théorique de Fredholm (Wireless telegraphy and Fredholm's theoretical method),
– La dynamique de l'électron et le principe de relativité (Dynamics of the electron and the principle of relativity).

Poincaré's lectures were held in May and June [31] 1908, and were edited in a series of five editions of the periodic *La Lumière Électrique*, which was at the time seen as a reference in electrotechnics and telegraphy. A very large public was reached, since readers of the periodic were added to the lectures' audience. In short, the published series, in chronological order, addressed the following subjects :

– Saturday 28 November 1908 (257) : L'émission d'ondes et l'amortissement (Wave emission and damping),
– Saturday 5 December 1908 (299) : Étude du champ dans le voisinage de l'antenne (Study of the field surrounding the antenna),
– Saturday 12 December 1908 (321) : Transmission des ondes et la diffraction (Transmission of waves and diffraction),
– Saturday 19 December 1908 (353) : La réception des signaux (Signal reception),
– Saturday 26 December 1908 (385) : Télégraphie dirigée. Oscillations entretenues (Directed telegraphy. Sustained oscillations).

It must be noted that each lecture was in the headlines of the publication. The editorial covered the contents and the main conclusions.

> "In these lectures, the author does not aim to create a complete theory of wireless telegraphy, but he imply intends to explain some mathematical theories likely to facilitate the understanding of these phenomena." (Poincaré 1908, 257)

Also it is noteworthy that these that these lectures brought as many points of view and theories as the original theoretical elements written by Poincaré. Until now, no traces of them have been found anywhere. The best example of this contribution addresses the question of sustained waves, which was the subject of the last lectures, cited in the editorial :

> "He then goes on to the study of sustained oscillations, and establishes four general equations, one of which is differential, in order to determine the stability condition of the regime, as well as the possibility conditions of the problem. From a practical standpoint,

31. From Lebon (1912, 67).

the simple inclusion of an arc in the circuit enables sustained oscillations, provided that a specific frequency is not exceeded, as shown in the calculation." (Poincaré, 1908, 385)

The same editorial mentioned the issue concerning the arc symmetry and the maintenance of asymmetry that would insure the existence of oscillations on all frequencies. Any reader of the periodic, including specialists in electrotechnics and the then-emerging wireless telegraphy, had therefore access to these writings expressed in perfectly comprehensible, clear and synthetic words.

### 1.4.1 Setting into equation the oscillations sustained by the singing arc

In his last lecture Henri Poincaré looked more specifically into the singing arc's device and the sustained oscillations it produced. The diagram of the circuit shown on Figure 1.16 is identical to Blondel's (1905b, 77) (see *supra* Figure 1.12, 22) :

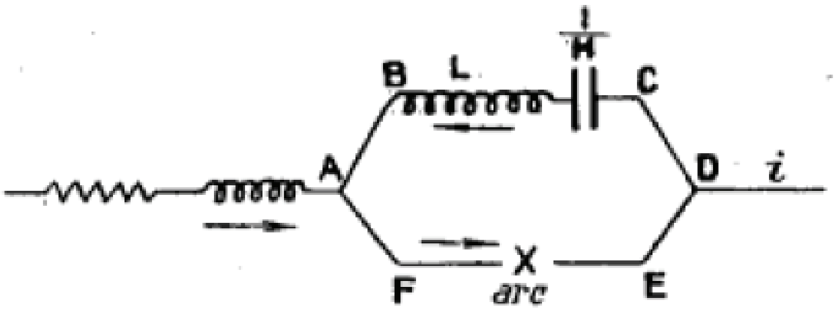


FIGURE 1.16 – Oscillations sustained by the singing arc, from Poincaré (1908, 390).

This circuit includes a "source of constant electromotive force $E$, a resistance and an inductor, and in parallel, an arc on one side, and an inductor and a capacitor on the other side." (Poincaré 1908, 390). He then wrote the very first equation for singing arc oscillations, calling $x$ the capacitor charge and $i$ the current in the external circuit. The current intensity in the branch including the capacitor, of capacitance $\frac{1}{H}$, is written as follows :

$$x' = \frac{dx}{dt}$$

Let $i_a$ be the current voltage in the arc, by applying Kirchhoff's (1824-1887) first law [32] and taking the current direction into account (see Fig. 1.17). Poincaré got the following result : $i = i_a - x'$. Therefore, the current in the arc is $i_a = i + x'$. By expressing, using Kirchhoff's second law [33], the voltage in the mesh ABCDEF, Poincaré established the second order nonlinear differential equation for the oscillations sustained by the singing arc

32. Nodal rule.
33. Mesh rule.

$$Lx'' + \rho x' + \phi(i + x') + Hx = 0 \tag{P$_1$}$$

## 1.4.2 The singing arc's electromotive force

He specifies that "considering $\rho x'$ is a term referring to the internal resistance of the inductor and other possible causes of damping, including the antenna radiation, $\phi(i + x')$ is the term due to the arc." (Poincaré 1908, 390). The latter term represents the electromotive force of the singing arc which should be related to the voltage running through it by an empirically determined relation (see supra Table 1.2, 16). This relation is indeterminable, making it impossible to integrate the equation (P1), and has been discredited due to the controversies linked with the presence of a counter-electromotive force in the arc and the difficulties inherent to experimentation, as stated by Paul Janet (1919). This did not seem to impede Henri Poincaré's research and he approached the problem as if it had already been resolved. In order to bypass this difficulty, he expresses the tension in the mesh AFED. A simplified version of the circuit is shown below (Fig. 1.17) to visualize the equation he found.

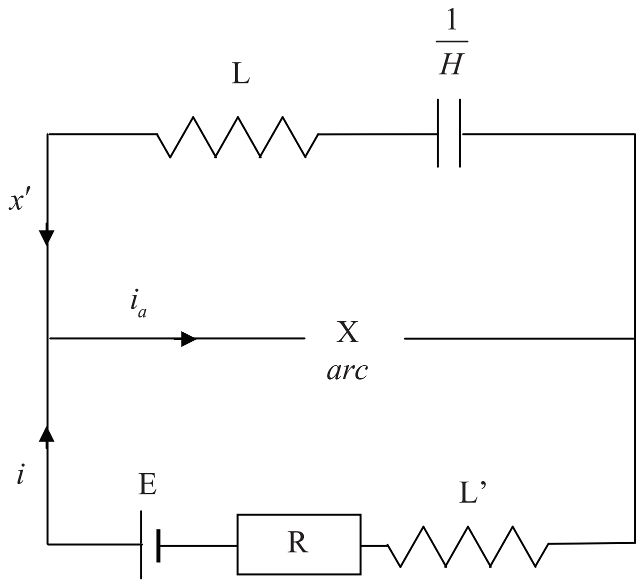


FIGURE 1.17 – Oscillations sustained by the *singing arc*, simplified version.

By neglecting the external inductor L', and equating the tension in the lower and middle branches of the simplified circuit (Fig. 1.17) he wrote : $E - Ri = \phi(i + x')$ he then deduced :

$$Ri + \phi(i + x') = E \tag{P$_2$}$$

He then explained that "if the function $\phi$ is assumed to be known, the equation (P$_2$) provides a relation between $i$ and $x'$ or between $i + x'$ and $x'$." (Poincaré 1908, 390). Indeed, it is easy to check that if we choose for this function that Poincaré (1908, 392) uses a few pages later and which is that of S.P. Thomson's [34] (see Table 1.2) : $\phi\left(i+x'\right)=\dfrac{a}{i+x'}+b$, we obtain the following equation :

$$Ri^2+\left(Rx'+b-E\right)i+a+\left(b-E\right)x'=0$$

Solving this second order equation for $i$ indeed provides a relation between $i$ and $x'$ but Poincaré's reasoning, probably based on the *implicit function theorem* enabled him to bypass all these calculations. If we assume as he did that the function $\phi$ is known, the equation ($P_2$) leads to a function F relating $i$ and $x'$. We will write out : $i=F\left(x'\right)$ and replace in the equation ($P_2$) thus obtaining :

$$\phi\left(i+x'\right)=E-RF\left(x'\right)=\theta\left(x'\right)$$

He therefore managed to replace $\phi$ in the equation ($P_1$) by $\theta\left(x'\right)$. The difficulty was lifted because the differential equation did not depend on only one variable $x$ anymore. He wrote it as follows :

$$Lx''+\rho x'+\theta\left(x'\right)+Hx=0 \qquad (P_3)$$

Poincaré thus created the very first incomplete equation modeling [35] of the oscillations occurring in the singing arc. It should be noted that this equation ($P_3$) corresponds exactly, in accordance to duality (see Appendix I.2), to the one established later by Blondel (1919b) for the triode, and Van der Pol (1920, 1926a, 1926b, 1926c, 1926d). Incidentally, the analogy between the oscillations sustained by the singing arc and the triode was brought to light by Paul Janet (1919).

### 1.4.3 Stability of the sustained oscillations and limit cycles

Poincaré therefore demonstrated more than twenty years before Andronov (1929a) that the stability of the equation ($P_3$) is related to the existence of a closed curve - a *limit cycle*. To achieve this he used the phase plane, which he introduced in his memoirs "Sur les Courbes définies par une équation différentielle" ("On the Curves defined by a differential equation") (Poincaré 1886, 168), by writing :

$$x'=\frac{dx}{dt}=y \quad ; \quad dt=\frac{dx}{y} \quad ; \quad x''=\frac{dy}{dt}=\frac{ydy}{dx}$$

The equation ($P_3$) becomes :

$$Ly\frac{dy}{dx}+\rho y+\theta\left(y\right)+Hx=0 \qquad (P_4)$$

Poincaré then provides the following representation :

34. It is interesting to note that Poincaré seemed to know about the latest hypotheses and theories regarding the arc, including Blondel's works (1897), since he did not choose Ayrton's relation (the most recent), but Thomson's, because it does not contain the notion of *c.e.m.f.* See Table 1.2.
35. The equation ($P_3$) is nonetheless incomplete, due to the lack of definition of the function *θ (x')*.

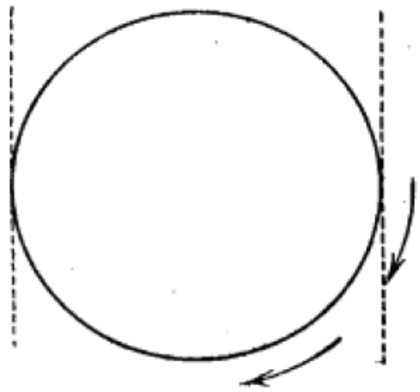

FIGURE 1.18 – Closed curve, from Poincaré (1908, 390).

It should be noted that this closed curve is represented in the phase plane $(x, y) = (x, \dot{x})$, *i.e.* the current-voltage phase plane [36] $(u_a, i_a)$ of the arc, and that it is only a metaphor of the actual solution, since Poincaré did not use any graphical integration method to find it. The only actual aim of this representation is to specify the direction of the trajectory curve which is a preliminary condition required to demonstrate the proof shown below.

> "Curves can be traced in a way that satisfies this differential equation, provided that the function $\theta$ is known. The sustained oscillations correspond to the closed curves if there are any. But any closed curve is not suitable ; it must meet specific stability conditions which we will study. Firstly, we see that, if $y = 0$, $\frac{dy}{dx}$ is infinite, the curve has vertical tangents. Besides, if $x$ decreases, $x'$ in other words $y$, is negative, therefore the curve must be traced in the direction of the arrow." (Poincaré 1908, 390).

From the equation ($P_4$), it can easily be found that :

$$\frac{dy}{dx} = -\frac{\rho y + \theta\,(y) + Hx}{Ly} = -\frac{\rho}{L} - \frac{\theta\,(y) + Hx}{Ly} \tag{P$_5$}$$

It can be deduced that when $y$ approaches zero the right-hand side of this equation becomes infinite. This closed curve therefore allows vertical tangents represented by dotted lines on Fig. 1.18. As for the direction, the reasoning is based on the fact that the derivative curve of an increasing function is negative. Therefore when $x$ decreases, $x'$ becomes negative. Yet, according to the equation given above : $x' = y$. This implies that y also becomes negative. This is only possible if the curve follows the direction of the arrow. It might seem surprising to look at the direction of the trajectory curve first. The reason is that the search for a condition provided by an inequality whose direction will

36. By rotating the $i$ and $u$ axes, we note that this is the same phase plane as that used by Blondel (1905a, 1681) (see page 23). It therefore appears that the hysteresis cycle corresponds exactly to Poincaré's closed curve, *i.e.* a limit cycle.

enable to govern the stability of sustained oscillations. The demonstration shown below uses Green's formula for integration, along a closed curve, which requires knowing the direction of the trajectory curves in order to define their orientation.

Poincaré then described a first version of the stability of sustained oscillations essentially based on the existence of a closed curve.

> "*Stability condition.* - Let us then consider another non-closed curve which satisfies the differential equation. It will be a kind of spiral growing indefinitely closer to the closed curve. If the closed curve represents a stable speed, by tracing the spiral in the direction of the arrow, we must be taken back to the closed curve, and this condition alone will enable the curve to represent a stable set of sustained waves, and solve the problem." (Poincaré 1908, 391)

In the *Note on the scientific Works of Henri Poincaré* which he wrote himself in 1886, he defines the concept of *limit cycle* :

> "This is how I call closed curves which fulfill the differential equation, for which the other curves defined by the same equation approach it asymptotically without ever reaching them." (Poincaré 1886, 30)

When comparing this excerpt with the *stability condition* explained in 1908, it appears clearly that the "closed curve", which represents the stable set of sustained waves, is actually a limit cycle as defined by Poincaré himself. The reason he did not write it out explicitly can however be addressed. It can be argued that firstly, this presentation was intended for engineers, and not mathematicians, and secondly, this terminology would have served no purpose [37]. An interesting comparison can be made between Poincaré's "stability condition" and the conclusion of paragraph 8, chapter V, of Andronov's book (1937), where he further explored the results he had obtained in 1928 (see *infra* Part II) and 1929 :

> "The existence of limit cycles in the description of the provided dynamic system's phase is a necessary and sufficient condition for the eventuality (provided the initial conditions are suitable) of self-oscillations in the system." (Andronov 1937, 293)

Poincaré then seems to avoid the "purely mathematical context" [38], in order to demonstrate that this problem was physically tangible. Beyond the connection between closed curves and sustained oscillations, Poincaré introduced the stability of the closed curve, *i.e.* of the limit cycle as an inequality.

> "*Possibility condition of the problem.* - Let us return to the equation ($P_4$). We multiply by $x'dt$, and integrate, over one period, the term L and the term $x$ leading to the integration of terms $x'$ and $x$, disappear, and we find :

37. The case of the center, which also constitutes a closed-curve solution, appears to have been excluded by Poincaré, insofar as it is a non-conservative system.
38. Diner (1992, 340).

$$\rho \int x'^2 dt + \int \theta\left(x'\right) x' dt = 0$$

Yet, the first term is certainly positive, and the function $\theta$ must therefore be thus :

$$\int \theta\left(x'\right) x' dt < 0.$$

Is it possible ?" (Poincaré 1908, 391)

The first integral equation is easy to establish. By following Poincaré's steps, we multiply the equation ($P_4$) by $x'dt$, taking the fact that $x'dt = dx$ into account, as shown in the equation above. We therefore find that :

$$\int Lydy + \int \rho x'^2 dt + \int \theta\left(x'\right) x' dt + \int Hxdx = 0$$

The first and last terms of this equation, which correspond to conservation of energy ($\frac{1}{2}Li^2 + \frac{1}{2}Hx^2$) cancel each other. The second integral inequation is deduced from the fact that the second quadratic term "is certainly positive" according to Poincaré (1908, 391). The *possibility condition of the problem* was therefore found. The relevance of a comparison between Poincaré's result and Andronov's (1929a) regarding the stability of the limit cycle therefore becomes apparent.

### 1.4.4 "Poincaré stability" and "Lyapunov stability"

Comparing Poincaré's *stability condition* (1908, 391) : $\int \theta\left(x'\right) x' dt < 0$ and Andronov's (1929a) requires the modification of the equation ($P_1$), and writing it as :

$$\begin{cases} \dfrac{dx}{dt} = Ly \\ \dfrac{dy}{dt} = -\rho y - \theta\left(y\right) - Hx \end{cases} \tag{$P_6$}$$

This equation ($P_6$) can be nondimensionalized by writing one the one hand $x \to \sqrt{\dfrac{L}{H}}x$ and $t \to \mu t$ with $\mu = \dfrac{1}{\sqrt{LH}} = \dfrac{1}{\omega}$, and on the other hand, neglecting the capacitor resistance $\rho$. We then find :

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - \mu\theta\left(y\right) \end{cases} \tag{$P_7$}$$

In his note (14 October 1929), Andronov (1929a, 560) considered the following set of differential equations :

$$\begin{cases} \dfrac{dx}{dt} = y + \mu f\left(x, y; \mu\right) \\ \dfrac{dy}{dt} = -x + \mu g\left(x, y; \mu\right) \end{cases} \tag{$A_4$}$$

where $\mu$ is a real parameter, which can be set as sufficiently small. He then specifies :

> "When $\mu = 0$, equations ($A_4$) have a solution $x = R\cos(t)$, $y = -R\sin(t)$ ; the solutions form, in the $xy$ plane, a family of circles. Following Poincaré's methods, it can be seen that for sufficiently small $\mu \neq 0$, the $xy$ plane contains only isolated closed curves, near to circles with radii defined by the equation
>
> $$\int_0^{2\pi} [f\left(R\cos\left(\xi\right), -R\sin\left(\xi\right); 0\right)\cos\left(\xi\right) - g\left(R\cos\left(\xi\right), -R\sin\left(\xi\right); 0\right)\sin\left(\xi\right)]d\xi = 0$$
>
> These closed curves correspond to stable, steady-state motion where the condition ($A_5$) is fulfilled :
>
> $$\int_0^{2\pi} \left[f'_x\left(R\cos\left(\xi\right), -R\sin\left(\xi\right); 0\right) + g'_y\left(R\cos\left(\xi\right), -R\sin\left(\xi\right); 0\right)\right]d\xi < 0 \tag{$A_5$}$$
>
> [...]" (Andronov 1929a, 561)

This result was actually the first draft of a theorem which was studied in a work titled "On Lyapunov stability" written by Andronov and Witt (1933), later formalized by Pontryagin (1934). By using Green's formula [39] and using a Cartesian coordinate system again, the stability condition ($A_5$) is written :

$$\int_\Gamma \left(f\left(x, y; \mu\right)\frac{dy}{dt} - g\left(x, y; \mu\right)\frac{dx}{dt}\right) dt < 0 \tag{$A_6$}$$

It therefore appears that Andronov's approach is perfectly identical to Poincaré's. A simple comparison between the two differential equations in Poincaré's phase plane can demonstrate this (see Table 1.3).

When writing : $x' = dx/dt = y$, $f\left(x, y; \mu\right) = 0$ and $g\left(x, y; \mu\right) = -\theta\left(y\right)$, it can be observed that Poincaré's system ($P_7$) corresponds exactly to Andronov's ($A_4$) and, Andronov's condition (1929a) ($A_6$) is in this case absolutely identical to Poincaré's *possibility condition of the problem* (1908).

39. Green's formula : $\int_\Gamma f\left(x, y\right) dy - g\left(x, y\right) dx = \iint_S \left(f'_x\left(x, y\right) + g'_y\left(x, y\right)\right) dxdy$

$$\int_{\Gamma} \left( f(x,y;\mu) \frac{dy}{dt} - g(x,y;\mu) \frac{dx}{dt} \right) dt < 0 \quad \Leftrightarrow \quad \int_{\Gamma} \theta(x') \, x' dt < 0$$

It therefore seems that Poincaré (1908) has not only established a connection between sustained oscillations and limit cycles, but has also demonstrated the limit cycle stability using a condition also found twenty years later by Andronov. Aside from the mathematical angle, this conclusion can also be reached by studying the bibliography of Andronov's article (1929a).

| Poincaré [1908a] | Andronov [1929a] |
| --- | --- |
| $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - \mu\theta(y) \end{cases}$ | $\begin{cases} \dfrac{dx}{dt} = y + \mu f(x,y;\mu) \\ \dfrac{dy}{dt} = -x + \mu g(x,y;\mu) \end{cases}$ |
| Voir Eq. ($P_7$) p. 21 | Voir Eq. ($A_4$) p. 22 |

TABLE 1.3 – Poincaré's (1908) and Andronov's (1929a) differential equation systems.

Despite all appearances, the actual connection with Poincaré's works is not the one found in his famous essay "Sur les Courbes" ("On curves") (1881-1886), more specifically in the chapter entitled "Limit cycles theory" (Poincaré 1882, 261), but is rather in relation with chapter III titled "Periodic solutions" in the "New Methods on Celestial Mechanics" (Poincaré 1892, 89). In this chapter Poincaré considered a system comparable to ($A_4$) which possessed a periodic solution for $\mu = 0$, and the following problem arose :

> "Under which conditions can we conclude that the equations still have periodic solutions for small values of $\mu$ ?" (Poincaré 1892, 81)

He then demonstrates that :

> "If the equations (1) depending on a parameter $\mu$ admit, for $\mu = 0$, a periodic solution with no characteristic exponents that are null, they will still admit a periodic solution for small values of $\mu$" (Poincaré 1892, 181)

The stability condition ($A_5$) corresponds to what Andronov and Witt (1930a, 1933) call in their articles the "Lyapunov stability" whilst referring to the chapter titled "Exposants caractéristiques'" ("characteristics exponents") of Poincaré (1892, 162). They define it in 1933 as follows :

> "In our case, one of the characteristic exponents is still null[40], since the equation (1) does not explicitly depend on time. The question of the "Lyapunov stability" is therefore whether the other characteristic exponents have negative real parts." (Andronov and Witt 1933, 373)

Poincaré's "possibility condition of the problem" (1908) and Andronov's condition of the "Lyapunov stability" (1929a) hence appear to be based on previous works by Poincaré : the negativity of one of the characteristic exponents. Andronov (1929a) seems to have found Poincaré's results (1908) independently with no previous knowledge of them. To prove the existence of a sustained oscillation regime Poincaré then demonstrated that the function $\phi$, which represents the electromotive force of the singing arc (see *supra*, 14) is decreasing. He achieved it by hypothesizing that "in the arc, the current $(i + x')$ always flows in the same direction and that the arc does not shut down." (Poincaré 1908, 391). He adds that we can "also assume that the direction of the current changes during an oscillation." (Poincaré 1908, 391). He then addressed a more practical aspect of the realization of the arc, and formulated the following condition for oscillation :

> "We can then see that, from the simple presence of an arc in the circuit, the function $\phi$ becomes decreasing. Therefore, from what has been previously stated, it becomes possible to have sustained oscillations." (Poincaré 1908, 391)

This sentence shows how far he managed to go in the interpretation and comprehension of the phenomenon. The decreasing in the function $\phi$ is closely linked to the concept of "negative resistance", which plays a crucial part in the sustaining of oscillations. After Henri Poincaré died in 1912, and during the First World War, a new device was developed which played a decisive role for the rest of the conflict in the field of communications : the *three-electrode tube*, or *triode*.

40. Andronov and Witt (1933) refer to Poincaré (1892, 180).

FIGURE 1.19 – Henri Poincaré.