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DIFFERENTIAL GEOMETRY AND MECHANICS: APPLICATIONS TO CHAOTIC DYNAMICAL SYSTEMS

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The aim of this article is to highlight the interest to apply Differential Geometry and Mechanics concepts to chaotic dynamical systems study. Thus, the local metric properties of curvature and torsion will directly provide the analytical expression of the slow manifold equation of slow-fast autonomous dynamical systems starting from kinematics variables (velocity, acceleration and over-acceleration or jerk).

The attractivity of the slow manifold will be characterized thanks to a criterion proposed by Henri Poincaré. Moreover, the specific use of acceleration will make it possible on the one hand to define slow and fast domains of the phase space and on the other hand, to provide an analytical equation of the slow manifold towards which all the trajectories converge. The attractive slow manifold constitutes a part of these dynamical systems attractor. So, in order to propose a description of the geometrical structure of attractor, a new manifold called singular manifold will be introduced. Various applications of this new approach to the models of Van der Pol, cubic-Chua, Lorenz, and Volterra–Gause are proposed.

Keywords: Differential geometry; curvature; torsion; slow-fast dynamics; strange attractors.

1. Introduction

There are various methods to determine the *slow manifold* analytical equation of slow-fast autonomous dynamical systems (S-FADS), or autonomous dynamical systems considered as slow-fast (CAS-FADS). A classical approach based on the works of Andronov [1966] led to the famous *singular approximation*. For $\varepsilon \neq 0$, another method called: *tangent linear system approximation*, developed by Rossetto *et al.* [1998], consists in using the presence of a "fast" eigenvalue in the functional jacobian matrix of a (S-FADS) or of a (CAS-FADS). Within the framework of application of the Tihonov's theorem [1952], this method uses the fact that in the vicinity of the *slow manifold* the eigenmode associated in the "fast" eigenvalue is evanescent. Thus, the *tangent linear system approximation* method, presented in the appendix, provides the *slow manifold* analytical equation of a dynamical system according to the "slow" eigenvectors of the *tangent linear system*, i.e. according to the "slow" eigenvalues. Nevertheless, according to the nature of the "slow" eigenvalues (real or complex conjugated) the plot of the *slow manifold* analytical equation may be difficult even impossible. Also to solve this problem it was necessary to make the *slow manifold* analytical equation independent of the "slow" eigenvalues. This could be carried out by multiplying the *slow manifold* analytical equation of a two-dimensional dynamical system by a "conjugated" equation, that of a three-dimensional dynamical system by two "conjugated" equations. In each case, the *slow manifold* analytical equation independent of the "slow" eigenvalues of the *tangent linear system* is presented in the appendix.

The new approach proposed in this article is based on the use of certain properties of *Differential Geometry* and *Mechanics*. Thus, the metric properties of *curvature* and *torsion* have provided a direct determination of the *slow manifold* analytical equation independently of the "slow" eigenvalues. It has been demonstrated that the equation thus obtained is completely identical to that which the *tangent linear system approximation* method provides.

The attractivity or repulsivity of the *slow manifold* could be characterized while using a criterion proposed by Henri Poincaré [1881] in a report entitled "Sur les courbes définies par une equation différentielle".

Moreover, the specific use of the instantaneous acceleration vector allowed a kinematic interpretation of the evolution of the *trajectory curves* in the vicinity of the *slow manifold* by defining the *slow* and *fast* domains of the phase space.

At last, two new manifolds called *singular* were introduced. It was shown that the first, the *singular approximation of acceleration*, constitutes a first approximation of the *slow manifold* analytical equation of a (S-FADS) and is completely equivalent to the equation provided by the method of the *successive approximations* developed by Rossetto [1986] while the second, the *singular manifold*, proposes an interpretation of the geometrical structure of these dynamical systems attractor.

2. Dynamical System, Slow-Fast Autonomous Dynamical System (S-FADS), Considered as Slow-Fast Autonomous Dynamical System (CAS-FADS)

The aim of this section is to recall definitions and properties of (S-FADS) and of (CAS-FADS).

2.1. *Dynamical system*

In the following we consider a system of differential equations defined in a compact E included in $\mathbb{R}$:

$$\frac{d\mathbf{X}}{dt} = \mathfrak{I}(\mathbf{X}) \tag{1}$$

with

$$\mathbf{X} = [x_1, x_2, \ldots, x_n]^t \in E \subset \mathbb{R}^n$$

and

$$\mathfrak{I}(\mathbf{X}) = [f_1(\mathbf{X}), f_2(\mathbf{X}), \ldots, f_n(\mathbf{X})]^t \in E \subset \mathbb{R}^n$$

The vector $\mathfrak{I}(\mathbf{X})$ defines a velocity vector field in E whose components f_i which are supposed to be continuous and infinitely differentiable with respect to all x_i and t, i.e. are C^∞ functions in E and with values included in $\mathbb{R}$, satisfy the assumptions of the Cauchy–Lipschitz theorem. For more details, see for example [Coddington & Levinson, 1955]. A solution of this system is an integral curve $\mathbf{X}(t)$ tangent to $\mathfrak{I}$ whose values define the *states* of the *dynamical system* described by Eq. (1). Since none of the components f_i of the velocity vector field depends here explicitly on time, the system is said to be *autonomous*.

Note: In certain applications, it would be supposed that the components f_i are C^r functions in E and with values in $\mathbb{R}$, with $r \geq n$.

2.2. *Slow-fast autonomous dynamical system (S-FADS)*

A (S-FADS) is a *dynamical system* defined under the same conditions as above but comprising a small multiplicative parameter ε in one or several components of its velocity vector field:

$$\frac{d\mathbf{X}}{dt} = \mathfrak{I}(\mathbf{X}) \tag{2}$$

with

$$\frac{d\mathbf{X}}{dt} = \left[\varepsilon\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right]^t \in E \subset \mathbb{R}^n$$

$$0 < \varepsilon \ll 1$$

The functional jacobian of a (S-FADS) defined by (2) has an eigenvalue called "fast", i.e. with a large real part on a large domain of the phase space. Thus, a "fast" eigenvalue is expressed like a polynomial of valuation -1 in ε and the eigenmode which is associated in this "fast" eigenvalue is said:

- "evanescent" if it is negative,
- "dominant" if it is positive.

The other eigenvalues called "slow" are expressed like a polynomial of valuation 0 in ε.

2.3. *Dynamical system considered as slow-fast (CAS-FADS)*

It has been shown [Rossetto *et al.*, 1998] that a dynamical system defined under the same conditions as (1) but without small multiplicative parameters in one of the components of its velocity vector field, and consequently without *singular approximation*, can be considered as *slow-fast* if its functional jacobian matrix has at least a "fast" eigenvalue, i.e. with a large real part on a large domain of the phase space.

3. New Approach of the Slow Manifold of Dynamical Systems

This approach consists in applying certain concepts of *Mechanics* and *Differential Geometry* to the study of dynamical systems (S-FADS or CAS-FADS). *Mechanics* will provide an interpretation of the behavior of the *trajectory curves*, integral of a (S-FADS) or of a (CAS-FADS), during the various phases of their motion in terms of *kinematics* variables: velocity and acceleration. The use of *Differential Geometry*, more particularly the local metric properties of *curvature* and *torsion*, will make it possible to directly determine the analytical equation of the *slow manifold* of (S-FADS) or of (CAS-FADS).

3.1. *Kinematics vector functions*

Since our proposed approach consists in using the *Mechanics* formalism, it is first necessary to define the *kinematics* variables needed for its development. Thus, we can associate the integral of the system (1) or (2) with the coordinates, i.e. with the position, of a moving point M at the instant t. This integral curve defined by the vector function $\mathbf{X}(t)$ of the scalar variable t represents the *trajectory curve* of the moving point M.

3.1.1. *Instantaneous velocity vector*

As the vector function $\mathbf{X}(t)$ of the scalar variable t represents the *trajectory* of M, the total derivative of $\mathbf{X}(t)$ is the vector function $\mathbf{V}(t)$ of the scalar variable t which represents the instantaneous velocity vector of the mobile M at the instant t; namely:

$$\mathbf{V}(t) = \frac{d\mathbf{X}}{dt} = \mathfrak{I}(\mathbf{X}) \tag{3}$$

The instantaneous velocity vector $\mathbf{V}(t)$ is supported by the tangent to the *trajectory curve*.

3.1.2. *Instantaneous acceleration vector*

As the instantaneous vector function $\mathbf{V}(t)$ of the scalar variable t represents the velocity vector of M, the total derivative of $\mathbf{V}(t)$ is the vector function $\boldsymbol{\gamma}(t)$ of the scalar variable t which represents the instantaneous acceleration vector of the mobile M at the instant t; namely:

$$\boldsymbol{\gamma}(t) = \frac{d\mathbf{V}}{dt} \tag{4}$$

Since the functions f_i are supposed to be C^∞ functions in a compact E included in $\mathbb{R}^n$, it is possible to calculate the total derivative of the vector field $\mathbf{V}(t)$ defined by (1) or (2). By using the derivatives of composite functions, we can write the derivative in the sense of Fréchet:

$$\frac{d\mathbf{V}}{dt} = \frac{d\mathfrak{I}}{d\mathbf{X}}\frac{d\mathbf{X}}{dt} \tag{5}$$

By noticing that $d\mathfrak{I}/d\mathbf{X}$ is the functional jacobian matrix J of the system (1) or (2), it follows from Eqs. (4) and (5) that we have the following equation which plays a very important role:

$$\boldsymbol{\gamma} = J\mathbf{V} \tag{6}$$

3.1.3. *Tangential and normal components of the instantaneous acceleration vector*

By making the use of the Frénet [1847] frame, i.e. a frame built starting from the *trajectory curve* $\mathbf{X}(t)$ directed towards the motion of the mobile M. Let us define $\boldsymbol{\tau}$ the unit tangent vector to the *trajectory curve* in M, $\boldsymbol{\nu}$ the unit normal vector, i.e. the principal normal in M directed towards the interior of the concavity of the curve and $\boldsymbol{\beta}$ the unit binormal vector to the *trajectory curve* in M so that the trihedron $(\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta})$ is direct. Since the instantaneous velocity vector $\mathbf{V}$ is tangent to any point M to the *trajectory curve* $\mathbf{X}(t)$, we can construct a unit tangent vector as follows:

$$\boldsymbol{\tau} = \frac{\mathbf{V}}{\|\mathbf{V}\|} \tag{7}$$

In the same manner, we can construct a unit binormal, as:

$$\boldsymbol{\beta} = \frac{\mathbf{V} \wedge \boldsymbol{\gamma}}{\|\mathbf{V} \wedge \boldsymbol{\gamma}\|} \tag{8}$$

and a unit normal vector, as:

$$\boldsymbol{\nu} = \boldsymbol{\beta} \wedge \boldsymbol{\tau} = \frac{\dot{\boldsymbol{\tau}}}{\|\dot{\boldsymbol{\tau}}\|} = \frac{\mathbf{V}^\perp}{\|\mathbf{V}^\perp\|} \tag{9}$$

with

$$\|\mathbf{V}\| = \|\mathbf{V}^\perp\| \tag{10}$$

where the vector $\mathbf{V}^{\perp}$ represents the normal vector to the instantaneous velocity vector $\mathbf{V}$ directed towards the interior of the concavity of the *trajectory curve* and where the dot $(\cdot)$ represents the derivative with respect to time. Thus, we can express the tangential and normal components of the instantaneous acceleration vector $\boldsymbol{\gamma}$ as:

$$\gamma_\tau = \frac{\boldsymbol{\gamma} \cdot \mathbf{V}}{\|\mathbf{V}\|} \tag{11}$$

$$\gamma_\nu = \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|}{\|\mathbf{V}\|} \tag{12}$$

By noticing that the variation of the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ can be written:

$$\frac{d\|\mathbf{V}\|}{dt} = \frac{\boldsymbol{\gamma} \cdot \mathbf{V}}{\|\mathbf{V}\|} \tag{13}$$

And while comparing Eqs. (11) and (12) we deduce that

$$\frac{d\|\mathbf{V}\|}{dt} = \gamma_\tau \tag{14}$$

Taking account of Eq. (10) and using the definitions of the scalar and vector products, the expressions of the tangential (11) and normal (12) components of the instantaneous acceleration vector $\boldsymbol{\gamma}$ can be finally written:

$$\gamma_\tau = \frac{\boldsymbol{\gamma} \cdot \mathbf{V}}{\|\mathbf{V}\|} = \frac{d\|\mathbf{V}\|}{dt} = \|\boldsymbol{\gamma}\|\cos(\widehat{\boldsymbol{\gamma}, \mathbf{V}}) \tag{15}$$

$$\gamma_\nu = \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|}{\|\mathbf{V}\|} = \|\boldsymbol{\gamma}\||\sin(\widehat{\boldsymbol{\gamma}, \mathbf{V}})| \tag{16}$$

Note: While using the *Lagrange identity*: $\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2 + (\boldsymbol{\gamma} \cdot \mathbf{V})^2 = \|\boldsymbol{\gamma}\|^2 \cdot \|\mathbf{V}\|^2$, one finds easily the norm of the instantaneous acceleration vector $\boldsymbol{\gamma}(t)$.

$$\begin{aligned}\|\boldsymbol{\gamma}\|^2 = \gamma_\tau^2 + \gamma_\nu^2 &= \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2}{\|\mathbf{V}\|^2} + \frac{(\boldsymbol{\gamma} \cdot \mathbf{V})^2}{\|\mathbf{V}\|^2} \\ &= \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2 + (\boldsymbol{\gamma} \cdot \mathbf{V})^2}{\|\mathbf{V}\|^2} = \|\boldsymbol{\gamma}\|^2\end{aligned}$$

3.2. *Trajectory curve properties*

In this approach the use of *Differential Geometry* will allow a study of the metric properties of the *trajectory curve*, i.e. *curvature* and *torsion* whose definitions are recalled in this section. One will find, for example, in [Delachet, 1964; Struik, 1934; Kreyzig, 1959] or [Gray, 2006] a presentation of these concepts.

3.2.1. *Parametrization of the trajectory curve*

The *trajectory curve* $\mathbf{X}(t)$ integral of the dynamical system defined by (1) or (2), is described by the motion of a current point M position which depends on a variable parameter: the time. This curve can also be defined by its parametric representation relative in a frame:

$$x_1 = F_1(t), \quad x_2 = F_2(t), \ldots, x_n = F_n(t)$$

where the F_i functions are continuous, C^∞ functions (or C^{r+1} according to the above assumptions) in E and with values in $\mathbb{R}$. Thus, the *trajectory curve* $\mathbf{X}(t)$ integral of the dynamical system defined by (1) or (2), can be considered as a *plane curve* or as a *space curve* having certain metric properties like *curvature* and *torsion* which will be defined below.

3.2.2. *Curvature of the trajectory curve*

Let us consider the *trajectory curve* $\mathbf{X}(t)$ having in M an instantaneous velocity vector $\mathbf{V}(t)$ and an instantaneous acceleration vector $\boldsymbol{\gamma}(t)$, the *curvature*, which expresses the rate of changes of the tangent to the *trajectory curve*, defined by:

$$\frac{1}{\Re} = \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|}{\|\mathbf{V}\|^3} = \frac{\gamma_\nu}{\|\mathbf{V}\|^2} \tag{17}$$

where $\Re$ represents the *radius of curvature*.

Note: The location of the points where the local *curvature* of the *trajectory curve* is null represents the location of the points of analytical inflexion, i.e. the location of the points where the normal component of the instantaneous acceleration vector $\boldsymbol{\gamma}(t)$ vanishes.

3.2.3. *Torsion of the trajectory curve*

Let us consider the *trajectory curve* $\mathbf{X}(t)$ having in M an instantaneous velocity vector $\mathbf{V}(t)$, an instantaneous acceleration vector $\boldsymbol{\gamma}(t)$, and an instantaneous over-acceleration vector $\dot{\boldsymbol{\gamma}}$, the *torsion*, which expresses the difference between the *trajectory curve* and a *plane* curve, defined by:

$$\frac{1}{\Im} = -\frac{\dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V})}{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2} \tag{18}$$

where $\Im$ represents the *radius of torsion*.

Note: A *trajectory curve* whose local *torsion* is null is a curve whose *osculating plane* is stationary. In this case, the *trajectory curve* is a plane curve.

3.3. *Application of these properties to the determination of the slow manifold analytical equation*

In this section it will be demonstrated that the use of the local metric properties of *curvature* and *torsion*, resulting from *Differential Geometry*, provide the analytical equation of the *slow manifold* of a (S-FADS) or a (CAS-FADS) of dimension two or three. Moreover, it will be established that the *slow manifold* analytical equation thus obtained is completely identical to that provided by the *tangent linear system approximation* method presented in the appendix.

3.3.1. *Slow manifold equation of a two-dimensional dynamical system*

Proposition 3.1. *The location of the points where the local curvature of the trajectory curves integral of a two-dimensional dynamical system defined by* (*1*) *or* (*2*) *is null, provides the slow manifold analytical equation associated to this system.*

Analytical Proof of Proposition 3.1. The vanishing condition of the *curvature* provides:

$$\frac{1}{\Re} = \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|}{\|\mathbf{V}\|^3} = 0 \Leftrightarrow \boldsymbol{\gamma} \wedge \mathbf{V} = \mathbf{0} \tag{19}$$

By using the expression (6), the coordinates of the acceleration vector are written:

$$\boldsymbol{\gamma}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} a\dot{x} + b\dot{y} \\ c\dot{x} + d\dot{y} \end{pmatrix}$$

The equation above is written:

$$c\dot{x}^2 - (a-d)\dot{x}\dot{y} - b\dot{y}^2 = 0$$

This equation is absolutely identical to Eq. (A.27) obtained by the *tangent linear system approximation* method. ■

Geometrical Proof of Proposition 3.1. The vanishing condition of the *curvature* provides:

$$\frac{1}{\Re} = \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|}{\|\mathbf{V}\|^3} = 0 \Leftrightarrow \boldsymbol{\gamma} \wedge \mathbf{V} = \mathbf{0}$$

The *tangent linear system approximation* makes it possible to write that:

$$\mathbf{V} = \alpha \mathbf{Y}_{\lambda_1} + \beta \mathbf{Y}_{\lambda_2} \approx \beta \mathbf{Y}_{\lambda_2}$$

While replacing in the expression (6) we obtain:

$$\boldsymbol{\gamma} = J\mathbf{V} = J(\beta \mathbf{Y}_{\lambda_2}) = \beta\lambda_2 \mathbf{Y}_{\lambda_2} = \lambda_2 \mathbf{V}$$

This shows that the instantaneous velocity and acceleration vectors are collinear, which results in:

$$\boldsymbol{\gamma} \wedge \mathbf{V} = \mathbf{0} \quad \blacksquare$$

3.3.2. *Slow manifold equation of a three-dimensional dynamical system*

Proposition 3.2. *The location of the points where the local torsion of the trajectory curves integral of a three-dimensional dynamical system defined by* (*1*) *or* (*2*) *is null, provides the slow manifold analytical equation associated to this system.*

Analytical Proof of Proposition 3.2. The vanishing condition of the *torsion* provides:

$$\frac{1}{\Im} = -\frac{\dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V})}{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2} = 0 \Leftrightarrow \dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V}) = 0 \tag{20}$$

The first corollary inherent in the *tangent linear system approximation* method implies to suppose that the functional jacobian matrix is stationary. That is to say

$$\frac{dJ}{dt} = 0$$

Derivative of the expression (6) provides:

$$\dot{\boldsymbol{\gamma}} = J\frac{d\mathbf{V}}{dt} + \frac{dJ}{dt}\mathbf{V} = J\boldsymbol{\gamma} + \frac{dJ}{dt}\mathbf{V}$$

$$= J^2\mathbf{V} + \frac{dJ}{dt}\mathbf{V} \approx J^2\mathbf{V}$$

The equation above is written as:

$$(J^2\mathbf{V}) \cdot (\boldsymbol{\gamma} \wedge \mathbf{V}) = 0$$

By developing this equation one finds in the long term Eq. (A.34) obtained by the *tangent linear system approximation* method. The two equations are thus absolutely identical. ■

Geometrical Proof of Proposition 3.2. The *tangent linear system approximation* makes it possible to write that:

$$\mathbf{V} = \alpha \mathbf{Y}_{\lambda_1} + \beta \mathbf{Y}_{\lambda_2} + \delta \mathbf{Y}_{\lambda_3} \approx \beta \mathbf{Y}_{\lambda_2} + \delta \mathbf{Y}_{\lambda_3}$$

While replacing in the expression (6) we obtain:

$$\boldsymbol{\gamma} = J\mathbf{V} = J(\beta \mathbf{Y}_{\lambda_2} + \delta \mathbf{Y}_{\lambda_3}) = \beta\lambda_2 \mathbf{Y}_{\lambda_2} + \delta\lambda_3 \mathbf{Y}_{\lambda_3}$$

According to what precedes, the over-acceleration vector is written as:

$$\dot{\boldsymbol{\gamma}} \approx J^2\mathbf{V}$$

While replacing the velocity by its expression we have:

$$\dot{\boldsymbol{\gamma}} \approx J^2(\beta \mathbf{Y}_{\lambda_2} + \delta \mathbf{Y}_{\lambda_3}) = \beta\lambda_2^2 \mathbf{Y}_{\lambda_2} + \delta\lambda_3^2 \mathbf{Y}_{\lambda_3}$$

Thus it is noticed that:

$$\begin{aligned}\mathbf{V} &= \beta \mathbf{Y}_{\lambda_2} + \delta \mathbf{Y}_{\lambda_3} \\ \boldsymbol{\gamma} &= \beta \lambda_2 \mathbf{Y}_{\lambda_2} + \delta \lambda_3 \mathbf{Y}_{\lambda_3} \\ \dot{\boldsymbol{\gamma}} &= \beta \lambda_2^2 \mathbf{Y}_{\lambda_2} + \delta \lambda_3^2 \mathbf{Y}_{\lambda_3}\end{aligned}$$

This demonstrates that the instantaneous velocity, acceleration and over-acceleration vectors are coplanar, which results in:

$$\dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V}) = 0$$

This equation represents the location of the points where *torsion* is null. The identity of the two methods is thus established. ■

Note: Main results of this study are summarized in Table 1 presented below. Abbreviations mean:

— T.L.S.A.: Tangent Linear System Approximation
— A.D.G.F.: Application of the Differential Geometry Formalism

In the first report entitled the "Courbes définies par une équation différentielle" Henri Poincaré [1881] proposed a criterion making it possible to characterize the attractivity or the repulsivity of a manifold. This criterion is recalled in the next section.

3.3.3. *Attractive, repulsive manifolds*

Proposition 3.3. *Let $\mathbf{X}(t)$ be a trajectory curve having in M an instantaneous velocity vector $\mathbf{V}(t)$ and let $(\mathcal{V})$ be a manifold (a curve in dimension two, a surface in dimension three) defined by the implicit equation $\phi = 0$ whose normal vector $\boldsymbol{\eta} = \boldsymbol{\nabla}\phi$ is directed towards the outside of the concavity of this manifold.*

- *If the scalar product between the instantaneous velocity vector $\mathbf{V}(t)$ and the normal vector $\boldsymbol{\eta} = \boldsymbol{\nabla}\phi$ is positive, the manifold is said attractive with respect to this trajectory curve*
- *If it is null, the trajectory curve is tangent to this manifold.*
- *If it is negative, the manifold is said repulsive.*

This scalar product which represents the total derivative of ϕ constitutes a new manifold $(\dot{\mathcal{V}})$ which is the envelope of the manifold $(\mathcal{V})$.

Table 1. Determination of the slow manifold analytical equation.

	T.L.S.A.	A.D.G.F.
Dimension 2	$\mathbf{V} \wedge \mathbf{Y}_{\lambda_2} = \mathbf{0}$	$\frac{1}{\Re} = \frac{\Vert\boldsymbol{\gamma} \wedge \mathbf{V}\Vert}{\Vert\mathbf{V}\Vert^3} = 0$
Dimension 3	$\mathbf{V} \cdot (\mathbf{Y}_{\lambda_2} \wedge \mathbf{Y}_{\lambda_3}) = 0$	$\frac{1}{\Im} = -\frac{\dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V})}{\Vert\boldsymbol{\gamma} \wedge \mathbf{V}\Vert^2} = 0$

Proof. Let us consider a *manifold* $(\mathcal{V})$ defined by the implicit equation $\phi(x, y, z) = 0$.

The normal vector directed towards the outside of the concavity of the *curvature* of this *manifold* is written as:

$$\boldsymbol{\eta} = \boldsymbol{\nabla}\phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix} \tag{21}$$

The instantaneous velocity vector of the *trajectory curve* is defined by (1):

$$\mathbf{V} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix}$$

The scalar product between these two vectors is written as:

$$\mathbf{V} \cdot \boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} \tag{22}$$

By noticing that Eq. (22) represents the total derivative of ϕ, the envelope theory makes it possible to state that the new *manifold* $(\dot{\mathcal{V}})$ defined by this total derivative constitutes the envelope of the *manifold* $(\mathcal{V})$ defined by the equation $\phi = 0$. The demonstration in dimension two of this Proposition results from what precedes. ■

3.3.4. *Slow, fast domains*

In the *Mechanics* formalism, the study of nature of motion of a mobile M consists in being interested in the variation of the Euclidean norm of its instantaneous velocity vector $\mathbf{V}$, i.e. in the tangential component γ_τ of its instantaneous acceleration vector $\boldsymbol{\gamma}$. The variation of the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ depends on the sign of the scalar product between the instantaneous velocity vector $\mathbf{V}$ and the instantaneous acceleration vector $\boldsymbol{\gamma}$, i.e. the angle formed by these two vectors. Thus if, $\boldsymbol{\gamma} \cdot \mathbf{V} > 0$, the variation of the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ is positive and the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ increases. The motion is accelerated, it is in its *fast* phase. If $\boldsymbol{\gamma} \cdot \mathbf{V} = 0$, the variation of the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ is null and the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ is constant. The motion is *uniform*, it is in a phase of transition between its *fast* and *slow* phases. Moreover, the instantaneous velocity vector $\mathbf{V}$ is perpendicular to the instantaneous acceleration vector $\boldsymbol{\gamma}$. If, $\boldsymbol{\gamma} \cdot \mathbf{V} < 0$, the variation of the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ is negative and the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ decreases. The motion is decelerated. It is in its *slow* phase.

Definition 3.1. The domain of the phase space in which the tangential component γ_τ of the instantaneous acceleration vector $\boldsymbol{\gamma}$ is negative, i.e. the domain in which the system is decelerating is called slow domain.

The domain of the phase space in which the tangential component γ_τ of the instantaneous acceleration vector $\boldsymbol{\gamma}$ is positive, i.e. the domain in which the system is accelerating is called fast domain.

Note: On the one hand, if the (S-FADS) studied comprises only one small multiplicative parameter ε in one of the components of its velocity vectors field, these two domains are complementary. The location of the points belonging to the domain of the phase space where the tangential component γ_τ of the instantaneous acceleration vector $\boldsymbol{\gamma}$ is cancelled, delimits the boundary between the *slow* and *fast* domains. On the other hand, the slow manifold of a (S-FADS) or a (CAS-FADS) necessarily belongs to the *slow* domain.

4. Singular Manifolds

The use of *Mechanics* made it possible to introduce a new manifold called *singular approximation of the acceleration* which provides an approximate equation of the *slow manifold* of a (S-FADS).

4.1. *Singular approximation of the acceleration*

The *singular perturbations theory* [Andronov *et al.*, 1966] have provided the zero order approximation in ε, i.e. the *singular approximation*, of the *slow manifold* equation associated in a (S-FADS) comprising a small multiplicative parameter ε in one of the components of its velocity vector field $\mathbf{V}$. In this section, it will be demonstrated that the *singular approximation* associated in the acceleration vector field $\boldsymbol{\gamma}$ constitutes the first-order approximation in ε of the *slow manifold* equation associated with a (S-FADS) comprising a small multiplicative parameter ε in one of the components of its velocity vector field $\mathbf{V}$ and consequently a small multiplicative parameter ε^2 in one of the components of its acceleration vector field $\boldsymbol{\gamma}$.

Proposition 4.1. *The manifold equation associated in the singular approximation of the instantaneous acceleration vector $\boldsymbol{\gamma}$ of a (S-FADS) constitutes the first-order approximation in ε of the slow manifold equation.*

Proof of Proposition 4.1 for Two-Dimensional (S-FADS). In dimension two, Proposition 3.1 results in a collinearity condition (19) between the instantaneous velocity vector $\mathbf{V}$ and the instantaneous acceleration vector $\boldsymbol{\gamma}$. While posing:

$$\frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{dy}{dt} = \dot{y} = g$$

The *slow manifold* equation of a (S-FADS) is written as:

$$\left(\frac{\partial g}{\partial x}\right)\dot{x}^2 - g\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)\dot{x} - \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)g^2 = 0 \tag{23}$$

This quadratic equation in $\dot{x}$ has the following discriminant:

$$\Delta = g^2\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)^2 + 4\left(\frac{\partial g}{\partial x}\right)\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)g^2 = 0$$

The Taylor series of its square root up to terms of order 1 in ε is written as:

$$\sqrt{\Delta} \approx \frac{1}{\varepsilon}\left|g\left(\frac{\partial f}{\partial x}\right)\right|\left\{1 + \frac{\varepsilon}{\left(\frac{\partial f}{\partial x}\right)^2}\left[2\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right) - \left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial f}{\partial x}\right)\right] + O(\varepsilon^2)\right\}$$

Taking into account what precedes, the solution of Eq. (23) is written as:

$$\dot{x} \approx -g\frac{\left(\dfrac{\partial f}{\partial y}\right)}{\left(\dfrac{\partial f}{\partial x}\right)} + O(\varepsilon) \tag{24}$$

This equation represents the second-order approximation in ε of the *slow manifold* equation associated with the *singular approximation*. According to Eq. (6), the instantaneous acceleration vector $\boldsymbol{\gamma}$ is written as:

$$\boldsymbol{\gamma} = \begin{pmatrix} \varepsilon\dfrac{d^2x}{dt^2} \\ \\ \dfrac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} \\ \\ \dfrac{\partial g}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial g}{\partial y}\dfrac{dy}{dt} \end{pmatrix}$$

The *singular approximation of the acceleration* provides the equation:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

While posing:

$$\frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{dy}{dt} = \dot{y} = g$$

We obtain:

$$\dot{x} = -g\frac{\left(\dfrac{\partial f}{\partial y}\right)}{\left(\dfrac{\partial f}{\partial x}\right)} \tag{25}$$

By comparing this expression with Eq. (24) which constitutes the second-order approximation in ε of the *slow manifold* equation, we deduce from Eq. (25) that it represents the first-order approximation in ε of the *slow manifold* equation. It has also been demonstrated that the *singular approximation of the acceleration* constitutes the first of the *successive approximations* developed in [Rossetto, 1986]. ■

Proof of Proposition 4.1 for Three-Dimensional (S-FADS). In dimension three, Proposition 3.2 results in a coplanarity condition (20) between the instantaneous velocity vector $\mathbf{V}$, the instantaneous acceleration vector $\boldsymbol{\gamma}$ and the instantaneous over-acceleration vector $\dot{\boldsymbol{\gamma}}$. The *slow manifold* equation of a (S-FADS) is written as:

$$\frac{d^3x}{dt^3}\begin{vmatrix} \dfrac{dy}{dt} & \dfrac{d^2y}{dt^2} \\ \dfrac{dz}{dt} & \dfrac{d^2z}{dt^2} \end{vmatrix} + \frac{d^3y}{dt^3}\begin{vmatrix} \dfrac{d^2x}{dt^2} & \dfrac{dx}{dt} \\ \dfrac{d^2z}{dt^2} & \dfrac{dz}{dt} \end{vmatrix} + \frac{d^3z}{dt^3}\begin{vmatrix} \dfrac{dx}{dt} & \dfrac{d^2x}{dt^2} \\ \dfrac{dy}{dt} & \dfrac{d^2y}{dt^2} \end{vmatrix} = 0 \tag{26}$$

In order to simplify, let us replace the three determinants by:

$$\Delta_1 = \begin{vmatrix} \dfrac{dy}{dt} & \dfrac{d^2y}{dt^2} \\ \dfrac{dz}{dt} & \dfrac{d^2z}{dt^2} \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} \dfrac{d^2x}{dt^2} & \dfrac{dx}{dt} \\ \dfrac{d^2z}{dt^2} & \dfrac{dz}{dt} \end{vmatrix};$$

$$\Delta_3 = \begin{vmatrix} \dfrac{dx}{dt} & \dfrac{d^2x}{dt^2} \\ \dfrac{dy}{dt} & \dfrac{d^2y}{dt^2} \end{vmatrix}$$

Equation (26) then will be written as:

$$(\dddot{x})\Delta_1 + (\dddot{y})\Delta_2 + (\dddot{z})\Delta_3 = 0 \tag{27}$$

While posing:

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y} = g, \quad \frac{dz}{dt} = \dot{z} = h$$

By dividing Eq. (27) by $(\dddot{z})$, we have

$$(\dot{x}\ddot{y} - \ddot{x}\dot{y}) + \frac{1}{(\dddot{z})}[(\dddot{x})\Delta_1 + (\dddot{y})\Delta_2] = 0 \tag{28}$$

First term of Eq. (28) is written as:

$$(\dot{x}\ddot{y} - \ddot{x}\dot{y}) = \left(\frac{\partial g}{\partial x}\right)\dot{x}^2 - g\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} - \frac{\partial g}{\partial z}\frac{h}{g}\right)\dot{x} - \frac{g^2}{\varepsilon}\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{h}{g}\right) \tag{29}$$

For homogeneity reasons let us pose:

$$g^2G = \frac{1}{(\dddot{z})}[(\dddot{x})\Delta_1 + (\dddot{y})\Delta_2]$$

Equation (28) is written:

$$\left(\frac{\partial g}{\partial x}\right)\dot{x}^2 - g\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} - \frac{\partial g}{\partial z}\frac{h}{g}\right)\dot{x} - \frac{g^2}{\varepsilon}\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{h}{g}\right) + g^2G = 0 \tag{30}$$

This quadratic equation in $\dot{x}$ has the following discriminant :

$$\Delta = g^2\left[\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \left(\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{h}{g}\right)\right]^2 + 4g^2\left(\frac{\partial g}{\partial x}\right)\left[\frac{1}{\varepsilon}\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{h}{g}\right) - G\right]$$

The Taylor series of its square root up to terms of order 1 in ε is written as:

$$\sqrt{\Delta} \approx \frac{1}{\varepsilon}\left|g\left(\frac{\partial f}{\partial x}\right)\right|\left\{1 + \frac{\varepsilon}{\left(\frac{\partial f}{\partial x}\right)^2}\left[2\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{h}{g}\right) - \left(\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{h}{g}\right)\left(\frac{\partial f}{\partial x}\right)\right] + O(\varepsilon^2)\right\}$$

Taking into account what precedes, the solution of Eq. (30) is written as:

$$\dot{x} \approx -g\frac{\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{h}{g}\right)}{\left(\frac{\partial f}{\partial x}\right)} + O(\varepsilon) \tag{31}$$

This equation represents the second-order approximation in ε of the *slow manifold* equation associated with the *singular approximation*. According to Eq. (6), the instantaneous acceleration vector $\boldsymbol{\gamma}$ is written as:

$$\boldsymbol{\gamma} = \begin{pmatrix} \varepsilon\frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \\ \frac{d^2z}{dt^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} \\ \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial z}\frac{dz}{dt} \end{pmatrix}$$

The *singular approximation of the acceleration* provides the equation:

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

While posing:

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y} = g \quad \text{and} \quad \frac{dz}{dt} = \dot{z} = h$$

We obtain:

$$\dot{x} = -g\frac{\left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{h}{g}\right)}{\left(\frac{\partial f}{\partial x}\right)} \tag{32}$$

By comparing this expression with Eq. (31) which constitutes the second-order approximation in ε of the *slow manifold* equation, we deduce from Eq. (32) that it represents the first-order approximation in ε of the *slow manifold* equation.

It has also been demonstrated that the *singular approximation of the acceleration* constitutes the first of the *successive approximations* developed in [Rossetto, 1986]. ■

The use of the criterion proposed by H. Poincaré (Proposition 3) made it possible to characterize the attractivity of the *slow manifold* of a (S-FADS) or a (CAS-FADS). Moreover, the presence in the phase space of an attractive *slow manifold*, in the vicinity of which the *trajectory curves* converge, constitutes a part of the attractor.

The *singular manifold* presented in the next section proposes a description of the geometrical structure of the attractor.

4.2. *Singular manifold*

The denomination of *singular manifold* comes from the fact that this manifold plays the same role with respect to the attractor as a singular point with respect to the *trajectory curve*.

Proposition 4.2. *The singular manifold is defined by the intersection of slow manifold of equation* $\phi = 0$ *and an unspecified Poincaré section* (Σ) *made in its vicinity. Thus, it represents the location of the points satisfying*:

$$\phi \cap \Sigma = 0 \tag{33}$$

This *manifold* of codimension one is a *submanifold* of the *slow manifold*.

In dimension two, the *singular manifold* is reduced to a point.

In dimension three, it is a "line" or more exactly a "curve".

The location of the points obtained by integration in a given time of initial conditions taken on this *manifold* constitutes a *submanifold* also belonging to the attractor generated by the dynamical system. The whole of these *manifolds* corresponding to various points of integration makes it possible to reconstitute the attractor by *deployment* of these *singular manifolds*.

The concept of *deployment* will be illustrated in Sec. 5.4.

5. Applications

5.1. *Van der Pol model*

The oscillator of B. Van der Pol [1926] is a second-order system with nonlinear frictions which can be written as:

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

The particular form of the friction which can be carried out by an electric circuit causes a decrease of the amplitude of the great oscillations and an increase of the small. There are various ways of writing the previous equation like a first-order system. One of them is:

$$\begin{cases} \dot{x} = \alpha\left(x + y - \dfrac{x^3}{3}\right) \\ \dot{y} = -\dfrac{x}{\alpha} \end{cases}$$

When α becomes very large, x becomes a "fast" variable and y a "slow" variable. In order to analyze the limit $\alpha \to \infty$, we introduce a small parameter $\varepsilon = 1/\alpha^2$ and a "slow time" $t' = t/\alpha = \sqrt{\varepsilon}t$. Thus, the system can be written as:

$$\mathbf{V}\begin{pmatrix} \varepsilon\dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \mathfrak{I}\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} x + y - \dfrac{x^3}{3} \\ -x \end{pmatrix} \tag{34}$$

with ε a positive real parameter

$$\varepsilon = 0.05$$

where the functions f and g are infinitely differentiable with respect to all x_i and t, i.e. C^∞ functions in a compact E are included in $\mathbb{R}^2$ and with values in $\mathbb{R}$. Moreover, the presence of a small multiplicative parameter ε in one of the components of its velocity vector field $\mathbf{V}$ ensures that the system (34) is a (S-FADS). We can thus apply the method described in Sec. 3, i.e. *Differential Geometry*. The instantaneous acceleration vector $\boldsymbol{\gamma}$ is written as:

$$\boldsymbol{\gamma}\begin{pmatrix} \dfrac{d^2x}{dt^2} \\ \dfrac{d^2y}{dt^2} \end{pmatrix} = \frac{d\mathfrak{I}}{dt}\begin{pmatrix} \dfrac{1}{\varepsilon}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt} - x^2\dfrac{dx}{dt}\right) \\ -\dfrac{dx}{dt} \end{pmatrix} \tag{35}$$

Proposition 3.1 leads to:

$$\frac{1}{\Re} = \frac{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|}{\|\mathbf{V}\|^3} = 0 \Leftrightarrow \boldsymbol{\gamma} \wedge \mathbf{V} = \mathbf{0} \Leftrightarrow \ddot{x}\dot{y} - \dot{x}\ddot{y} = 0$$

We obtain the following implicit equation:

$$\frac{1}{9\varepsilon^2}[9y^2 + (9x + 3x^3)y + 6x^4 - 2x^6 + 9x^2\varepsilon] = 0 \tag{36}$$

Since this equation is quadratic in y, we can solve it in order to plot y according to x.

$$y_{1,2} = -\frac{x^3}{6} - \frac{x}{2} \pm \frac{x}{2}\sqrt{x^4 - 2x^2 + 1 - 4\varepsilon} \tag{37}$$

Figure 1 shows the plot of the *slow manifold* equation (37) of the Van der Pol system with $\varepsilon = 0.05$ by using Proposition 3.1, i.e. the collinearity condition between the instantaneous velocity vector $\mathbf{V}$ and the instantaneous acceleration vector $\boldsymbol{\gamma}$, i.e. the location of the points where the *curvature* of the *trajectory curves* is cancelled. Moreover, Definition 1 makes it possible to delimit the area of the phase plane in which, the scalar product between the instantaneous velocity vector $\mathbf{V}$ and the instantaneous acceleration vector $\boldsymbol{\gamma}$ is negative, i.e. where the tangential component γ_τ of its instantaneous acceleration vector $\boldsymbol{\gamma}$ is negative. We can thus graphically distinguish the *slow* domain of the *fast* domain (in blue), i.e. the domain of stability of the *trajectories*.

The blue part of Fig. 1 corresponds to the domain where the variation of the Euclidean norm of the instantaneous velocity vector $\mathbf{V}$ is positive, i.e. where the tangential component of the

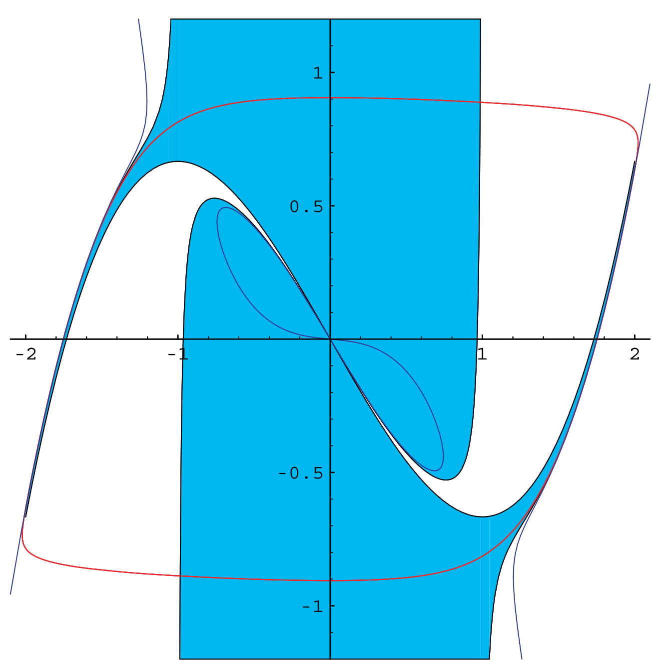

Fig. 1. Slow manifold (in blue) and stable domain of the Van der Pol system (in white).

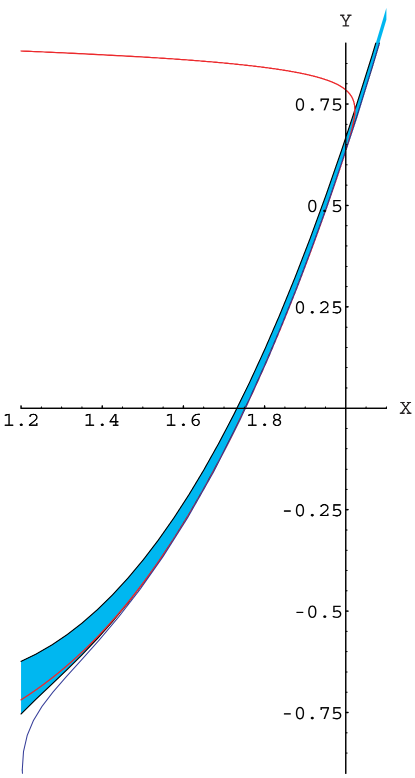

Fig. 2. Zoom of the stable part of the B. Van der Pol system.

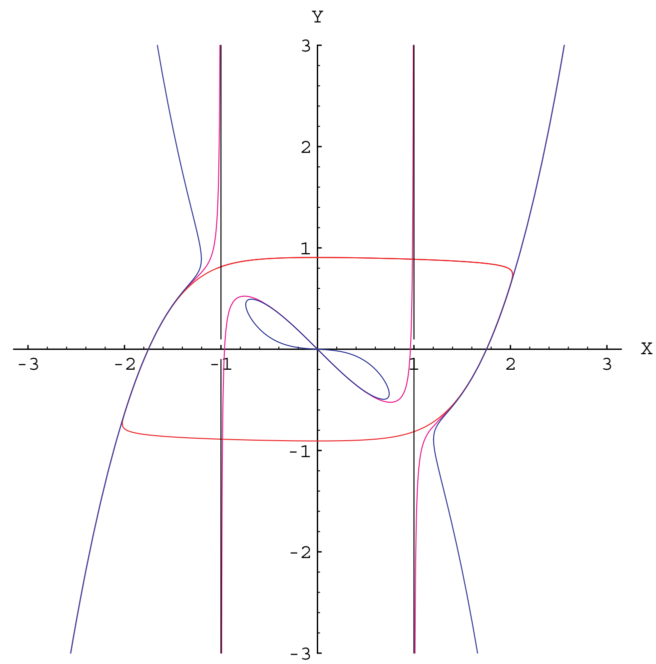

Fig. 3. Singular approximation of the acceleration of the B. Van der Pol system (in magenta).

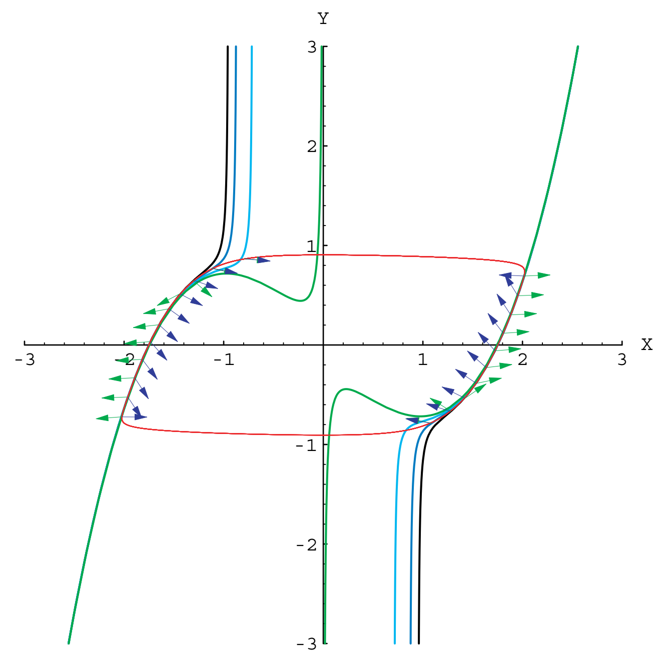

Fig. 4. Isoclines of the acceleration vector of the B. Van der Pol system.

instantaneous acceleration vector $\boldsymbol{\gamma}$, is positive. Let us take note that, as soon as the *trajectory curve*, initially outside this domain, enters inside, it leaves the *slow manifold* to reach the *fast foliation*.

The *slow manifold* equation provided by Proposition 4.1 leads to the following implicit equation:

$$\frac{1}{3\varepsilon^2}[3x - 4x^3 + x^5 + (3 - 3x^2)y - 3x\varepsilon] = 0 \quad (38)$$

Starting from this equation we can plot y according to x:

$$y = \frac{x^5 - 4x^3 + 3x(1 - \varepsilon)}{3(-1 + x^2)} \quad (39)$$

Figure 3 shows the plot of the slow manifold equation (39) of the Van der Pol system with $\varepsilon = 0.05$ by using Proposition 4.1, i.e. the *singular approximation* of the instantaneous acceleration vector $\boldsymbol{\gamma}$ in magenta. Blue curve represents the *slow manifold* equation (37) provided by Proposition 3.1.

In order to illustrate the principle of the method presented above, we have plotted in Fig. 4 the isoclines of acceleration for various values: 0.5, 0.2, 0.1, 0.05.

The very large variation rate of the acceleration in the vicinity of the *slow manifold* can be noticed in Fig. 4. Some isoclines of the acceleration vector which tend to the *slow manifold* defined by Proposition 3.1 are plotted.

5.2. *Chua model*

The Chua circuit [1986] is a relaxation oscillator with a cubic nonlinear characteristic elaborated from a circuit comprising a harmonic oscillator for which the operation is based on a field-effect transistor, coupled to a relaxation-oscillator composed of a tunnel diode. The modeling of the circuit uses a capacity which will prevent abrupt voltage drops and makes it possible to describe the fast motion of this oscillator by the following equations which constitute a *slow-fast* system.

$$\mathbf{V} = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\varepsilon}\left(z - \dfrac{44}{3}x^3 - \dfrac{41}{2}x^2 - \mu x\right) \\ -z \\ -0.7x + y + 0.24z \end{pmatrix} \tag{40}$$

with ε and μ are real parameters

$$\varepsilon = 0.01$$
$$\mu = 6.94$$

where the functions f, g and h are infinitely differentiable with respect to all x_i, and t, i.e. C^∞ functions in a compact E are included in $\mathbb{R}^3$ and with values in $\mathbb{R}$. Moreover, the presence of a small multiplicative parameter ε in one of the components of its instantaneous velocity vector $\mathbf{V}$ ensures that the system (40) is a (S-FADS). We can thus apply the method described in Sec. 3, i.e. *Differential Geometry*. In dimension three, the *slow manifold* equation is provided by Proposition 3.2, i.e. the vanishing condition of the *torsion*:

$$\frac{1}{\Im} = -\frac{\dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V})}{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2} = 0 \Leftrightarrow \dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V}) = 0$$

Within the framework of the *tangent linear system approximation*, Corollary 1 leads to Eq. (34). Using Mathematica® Fig. 5 shows a plot of the phase portrait of Chua model and its *slow manifold.*

Without the framework of the *tangent linear system approximation*, i.e. considering that the functional jacobian varies with time, Proposition 3.2 provides a surface equation which represents the location of points where *torsion* is cancelled, i.e. the location of points where the *osculating plane* is stationary and where the *slow*

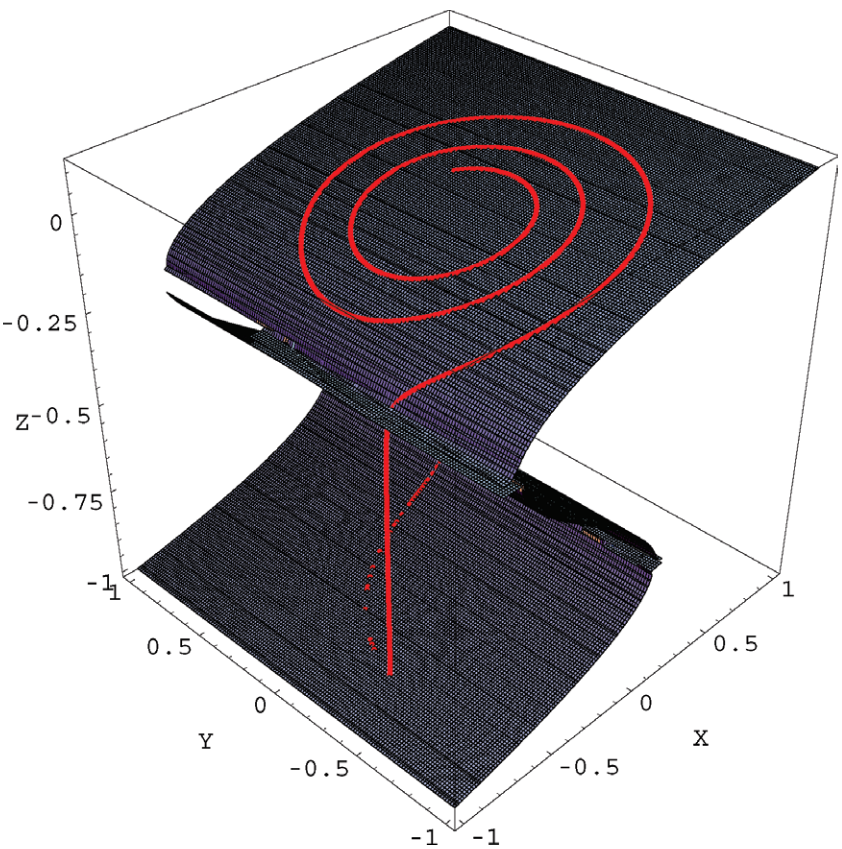

Fig. 5. Slow manifold of Chua model.

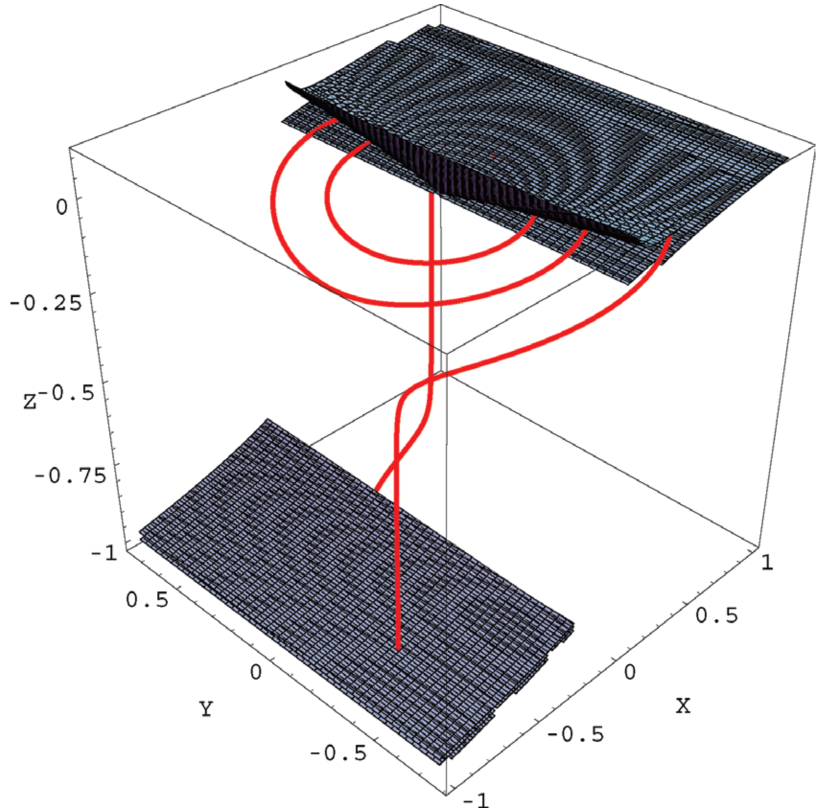

Fig. 6. Attractive part of the *slow manifold* of Chua model.

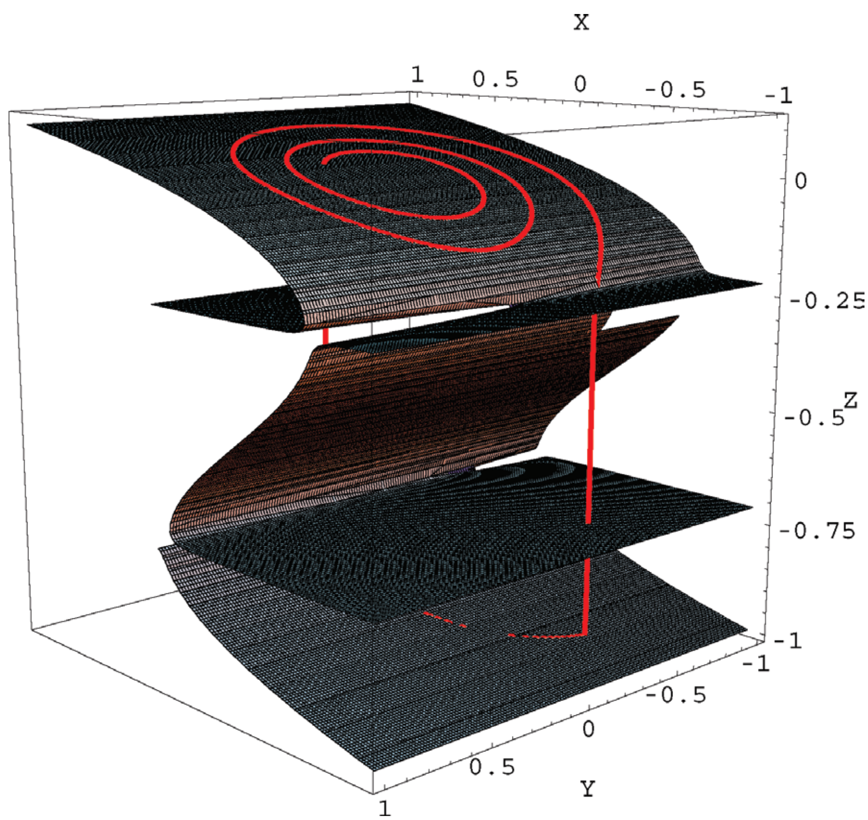

Fig. 7. Singular approximation of the acceleration of Chua model.

manifold is attractive. Thus, the attractive part of the *slow manifold* of Chua model is plotted in Fig. 6.

We deduce, according to Proposition 3.3, that the location of the points where the *torsion* is negative corresponds to the attractive parts of the *slow manifold*. Thus, the attractive part of the *slow manifold* of the Chua model is plotted in Fig. 6.

Slow manifold equation provided by Proposition 4.1 leads to the following implicit equation:

$$\begin{aligned}\frac{1}{6\varepsilon^2}(&5043x^3 + 9020x^4 + 3872x^5 - 246xz \\ &- 264x^2z - 4.2x\varepsilon + 6y\varepsilon + 1.44z\varepsilon \\ &+ 369x^2\mu + 352x^3\mu - 6z\mu + 6x\mu^2) = 0\end{aligned}$$

The surface plotted in Fig. 7 constitutes a quite good approximation of the *slow manifold* of this model.

6. Lorenz Model

The purpose of the model established by Edward Lorenz [1963] was in the beginning to analyze the impredictible behavior of weather. After having developed nonlinear partial derivative equations starting from the thermal equation and Navier–Stokes equations, Lorenz truncated them to retain only three modes. The most widespread form of the Lorenz model is as follows:

$$\mathbf{V} = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \sigma(y-x) \\ -xz + rx - y \\ xy - \beta z \end{pmatrix} \tag{41}$$

with σ, r and β are real parameters: $\sigma = 10$, $\beta = 8/3$, $r = 28$, where the functions f, g and h are infinitely differentiable with respect to all x_i and t, i.e. C^∞ functions in a compact E are included in $\mathbb{R}^3$ and with values in $\mathbb{R}$. Although this model has no *singular approximation*, it can be considered as a (S-FADS), according to Sec. 1.3, because it has been numerically checked [Rossetto *et al.*, 1998] that its functional jacobian matrix possesses at least a large and negative real eigenvalue in a large domain of the phase space. Thus, we can apply the method

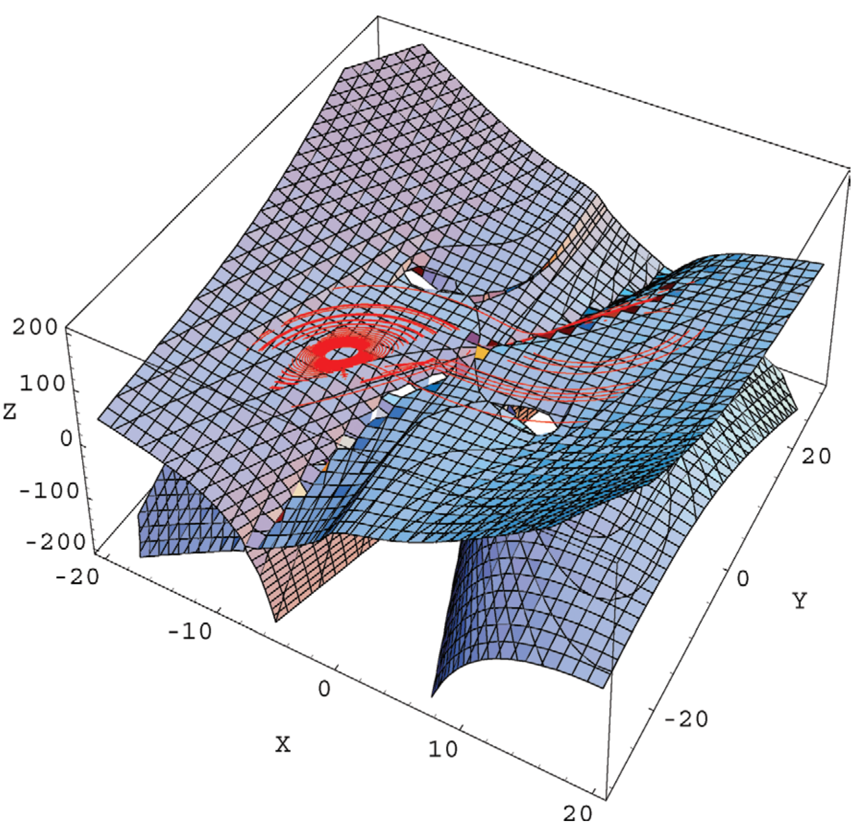

Fig. 8. Slow manifold of Lorenz model.

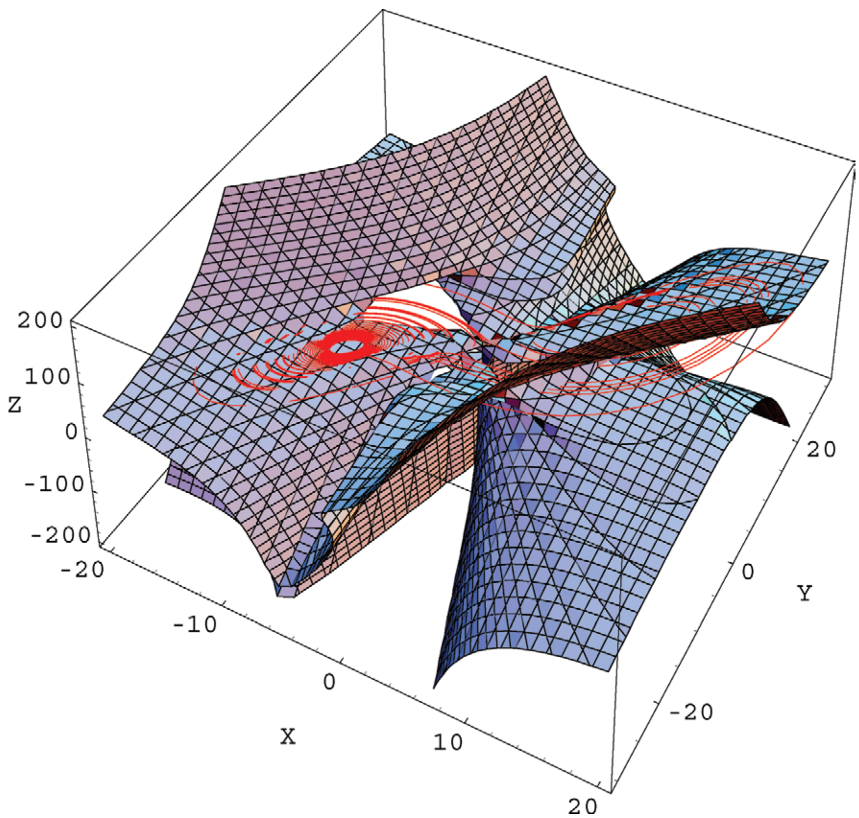

Fig. 9. Attractive part of the *slow manifold* of Lorenz model.

described in Sec. 3, i.e. *Differential Geometry*. In dimension three, the *slow manifold* equation is provided by Proposition 3.2, i.e. the vanishing condition of the *torsion*:

$$\frac{1}{\Im} = -\frac{\dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V})}{\|\boldsymbol{\gamma} \wedge \mathbf{V}\|^2} = 0 \Leftrightarrow \dot{\boldsymbol{\gamma}} \cdot (\boldsymbol{\gamma} \wedge \mathbf{V}) = 0$$

Within the framework of the *tangent linear system approximation*, Corollary 1 leads to Eq. (34). Using Mathematica® Fig. 8 shows the plot of the phase portrait of Lorenz model and its *slow manifold.*

Without the framework of the *tangent linear system approximation*, i.e. considering that the functional jacobian varies with time, Proposition 3.2 provides a surface equation which represents the location of the points where *torsion* is cancelled, i.e. the location of the points where the *osculating plane* is stationary and where the *slow manifold* is attractive. Thus, the attractive part of the *slow manifold* of Lorenz model is plotted in Fig. 9.

We deduce, according to Proposition 3.3, that the location of the points where the *torsion* is negative corresponds to the attractive parts of the *slow manifold*. Thus the attractive part of the *slow manifold* of the Lorenz model is plotted in Fig. 9.

7. Volterra–Gause Model

Let us consider the model elaborated by Ginoux *et al.* [2005] for three species interacting in a predator–prey mode.

$$\mathbf{V} = \begin{pmatrix} \dfrac{dx}{dt} \\[2mm] \dfrac{dy}{dt} \\[2mm] \dfrac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\xi}(x(1-x) - x^{\frac{1}{2}}y) \\ -\delta_1 y + x^{\frac{1}{2}}y - y^{\frac{1}{2}}z \\ \varepsilon z(y^{\frac{1}{2}} - \delta_2) \end{pmatrix} \tag{42}$$

with ξ, ε, δ_1 and δ_2 are real parameters: $\xi = 0.866$, $\varepsilon = 1.428$, $\delta_1 = 0.577$, $\delta_2 = 0.376$.

And where the functions f, g and h are infinitely differentiable with respect to all x_i, and t, i.e. C^∞ functions in a compact E are included in $\mathbb{R}^3$ and with values in $\mathbb{R}$.

This model consisting of a prey, a predator and top-predator has been named Volterra–Gause because it combines the original model of

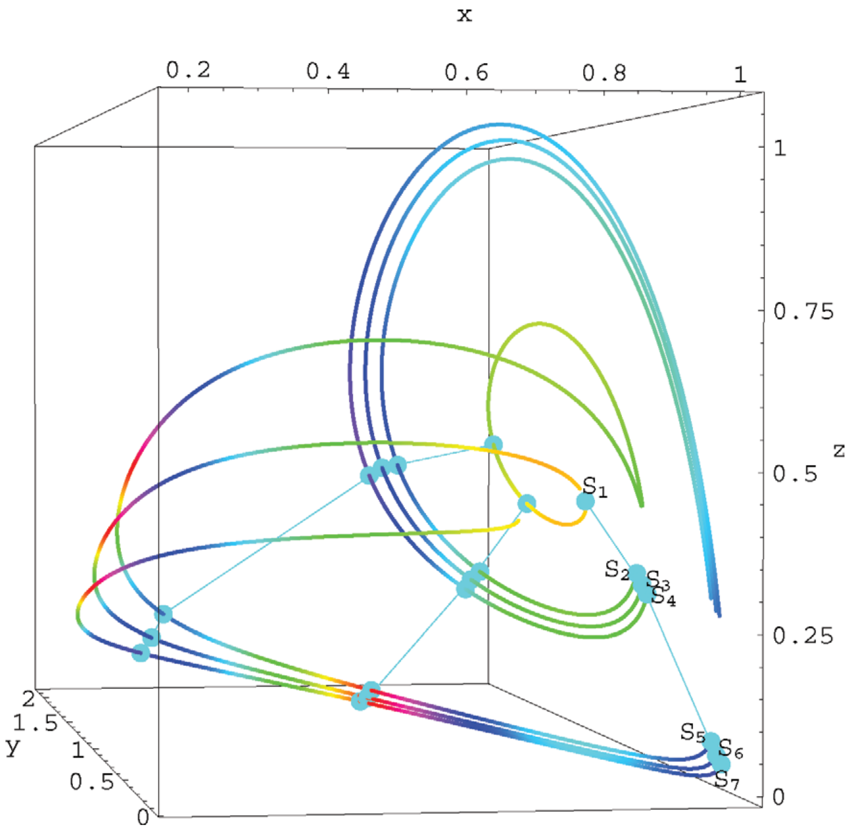

Fig. 10. Deployment of the *singular manifold* of the Volterra–Gause model.

V. Volterra [1926] incorporating a logisitic limitation of P. F. Verhulst [1838] type on the growth of the prey and a limitation of G. F. Gause [1935] type on the intensity of the predations of the predator on the prey and top-predator on the predator. The equations (42) are dimensionless, remarks and details about the changes of variables and the parameters have been extensively presented in [Ginoux *et al.*, 2005]. Moreover, the presence of a small multiplicative parameter ξ in one of the components of its instantaneous velocity vector $\mathbf{V}$ ensures that the system (42) is a (S-FADS). So, the method described in Sec. 3, i.e. *Differential Geometry* would have provided the *slow manifold* equation thanks to Proposition 3.2. But, this model exhibits a chaotic attractor in the snail shell shape and the use of the algorithm developed by Wolf *et al.* [1985] have made it possible to compute what can be regarded as its Lyapunov exponents: $(+0.035, 0.000, -0.628)$. Then, the Kaplan–Yorke [1983] conjecture provided the following Lyapunov dimension: 2.06. So, the fractal dimension of this chaotic attractor is close to that of a surface. The *singular manifold* makes it possible to account for the evolution of the *trajectory curves* on the surface generated by this attractor. Indeed, the location of the points of intersection of the *slow manifold* with a Poincaré section carried out in its vicinity constitutes a "line" or more exactly a "curve". Then by using numerical integration this "curve" (resp. "line") is deployed through the phase space and its deployment reconstitutes the attractor shape. The result is plotted in Fig. 10.

8. Discussion

Considering the *trajectory curves* integral of dynamical systems as *plane* or *space* curves evolving in the phase space, it has been demonstrated in this work that the local metric properties of *curvature* and *torsion* of these *trajectory curves* make it possible to directly provide the analytical equation of the *slow manifold* of dynamical systems (S-FADS or CAS-FADS) according to kinematics variables. The *slow manifold* analytical equation is thus given by the following:

- vanishing condition of the *curvature* in dimension two,

- vanishing condition of the *torsion* in dimension three.

Thus, the use of *Differential Geometry* concepts has made it possible for the analytical equation of the *slow manifold* to be completely independent of the "slow" eigenvectors of the functional jacobian of the *tangent linear system*, and it was demonstrated that the equation thus obtained is completely identical to that providing the *tangent linear system approximation* method [Rossetto *et al.*, 1998] presented below in the appendix. So characterization of its attractivity is possible while using a criterion proposed by Henri Poincaré [1881] in his report entitled "Sur les courbes définies par une equation différentielle".

Moreover, the specific use of the instantaneous acceleration vector, inherent in *Mechanics*, allows on the one hand a kinematics interpretation of the evolution of the *trajectory curves* in the vicinity of the *slow manifold* by defining the *slow* and *fast* domains of the phase space and on the other hand, to approach the analytical equation of the *slow manifold* thanks to the *singular approximation of acceleration*. The equation thus obtained is completely identical to that which provides the *successive approximations* method [Rossetto, 1986]. Thus, it has been established that the presence in the phase space of an attractive *slow manifold*, in the vicinity of which the *trajectory curves* converge, defines part of the attractor. So, in order to propose a qualitative description of the geometrical structure of attractor a new manifold called *singular* has been introduced.

Various applications to the models of Van der Pol, cubic-Chua, Lorenz and Volterra–Gause have made it possible to illustrate the practical interest of this new approach for the dynamical systems (S-FADS or CAS-FADS) study.

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References

Andronov, A. A., Khaikin, S. E. & Vitt, A. A. [1966] *Theory of Oscillators* (Pergamon Press, Oxford).

Chlouverakis, K. E. & Sprott, J. C. [2004] "A comparison of correlation and Lyapunov dimensions," *Physica D* **200**, 156–164.

Chua, L. O., Komuro, M. & Matsumoto, T. [1986] "The double scroll family," *IEEE Trans. Circuits Syst.* **33**, 1072–1097.

Coddington, E. A. & Levinson, N. [1955] *Theory of Ordinary Differential Equations* (MacGraw Hill, NY).

Delachet, A. [1964] *La Géométrie Différentielle*, "Que sais-je", n°1104 (PUF, Paris).

Frenet, F. [1852] "Sur les courbes à double courbure," Thèse Toulouse, 1847. Résumé dans *J. de Math.*, 17.

Gause, G. F. [1935] *The Struggle for Existence* (Williams and Wilkins, Baltimore).

Ginoux, J. M., Rossetto, B. & Jamet, J. L. [2005] "Chaos in a three-dimensional Volterra–Gause model of predator–prey type," *Int. J. Bifurcation and Chaos* **5**, 1689–1708.

Gray, A., Salamon, S. & Abbena, E. [2006] *Modern Differential Geometry of Curves and Surfaces with Mathematica* (Chapman & Hall/CRC).

Kaplan, J. & Yorke, J. A. [1979] "Chaotic behavior of multidimensional difference equations, in functional differential equations and approximation of fixed points," *Lecture Notes in Mathematics* **730**, 204–227.

Kreyszig, E. [1991] *Differential Geometry* (Dover, NY).

Lorenz, E. N. [1963] "Deterministic non-periodic flows," *J. Atmos. Sci.* **20**, 130–141.

Poincaré, H. [1881] "Sur les courbes définies par une équation différentielle," *J. Math. Pures et Appl.*, Série III **7**, 375–422.

Poincaré, H. [1882] "Sur les courbes définies par une équation différentielle," *J. Math. Pures et Appl.*, Série III **8**, 251–296.

Poincaré, H. [1885] "Sur les courbes définies par une équation différentielle," *J. Math. Pures et Appl.*, Série IV **1**, 167–244.

Poincaré, H. [1886] "Sur les courbes définies par une équation différentielle," *J. Math. Pures et Appl.*, Série IV **2**, 151–217.

Rossetto, B. [1986] "Singular approximation of chaotic slow-fast dynamical systems," *Lecture Notes in Physics* **278**, 12–14.

Rossetto, B., Lenzini, T., Ramdani, S. & Suchey, G. [1998] "Slow-fast autonomous dynamical systems," *Int. J. Bifurcation and Chaos* **8**, 2135–2145.

Struik, D. J. [1988] *Lecture on Classical Differential Geometry* (Dover, NY).

Tihonov, A. N. [1952] "Systems of differential equations containing small parameters in the derivatives," *Mat. Sbornik N.S.* **31**, 575–586.

Van der Pol, B. [1926] "On 'relaxation-oscillations'," *Phil. Mag.* **7**, 978–992.

Verhulst, P. F. [1838] "Notice sur la loi que suit la population dans son accroissement," *Corresp. Math. Phys.* **X**, 113–121.

Volterra, V. [1926] "Variazioni e fluttuazioni del numero d'individui in specie animali conviventi," *Mem. Acad. Lincei III* **6**, 31–113.
Volterra, V. [1931] *Leçons sur la Théorie Mathématique de la Lutte pour la Vie*, Gauthier-Villars, Paris.
Wolf, A., Swift, J. B., Swinney, H. L. & Vastano, J. A. [1985] "Determining Lyapunov exponents from a time series," *Physica D* **16**, 285–317.

Appendix

Formalization of the *Tangent Linear System Approximation* Method

The aim of this appendix is to demonstrate that the approach developed in this work generalizes the *tangent linear system approximation* method [Rossetto *et al.*, 1998]. After having pointed out the necessary assumptions to the application of this method and the corollaries which result from this, two conditions (of collinearity/coplanarity and orthogonality) providing the analytical equation of the *slow manifold* of a dynamical system defined by (1) or (2) will be presented in a formal way. Equivalence between these two conditions will then be established. Lastly, while using the sum and the product (also the square of the sum and the product) of the eigenvalues of the functional jacobian of the *tangent linear system*, the equation of the *slow manifold* generated by these two conditions will be made independent of these eigenvalues and will be expressed according to the elements of the functional jacobian matrix of the *tangent linear system*. It will be thus demonstrated that this analytical equation of the *slow manifold* is completely identical to that provided by the Propositions 3.1 and 3.2 developed in this article.

Assumptions

The application of the *tangent linear system approximation* method requires that the dynamical system defined by (1) or (2) satisfies the following assumptions:

(H$_1$) The components f_i, of the velocity vector field $\mathfrak{I}(\mathbf{X})$ defined in E are continuous, C^∞ functions in E and with values included in $\mathbb{R}$.

(H$_2$) The dynamical system defined by (1) or (2) satisfies the *nonlinear part condition* [Rossetto *et al.*, 1998], i.e. the influence of the nonlinear part of the Taylor series of the velocity vector field $\mathfrak{I}(\mathbf{X})$ of this system is overshadowed by the fast dynamics of the linear part.

$$\mathfrak{I}(\mathbf{X}) = \mathfrak{I}(\mathbf{X}_0) + (\mathbf{X}-\mathbf{X}_0)\left.\frac{d\mathfrak{I}(\mathbf{X})}{d\mathbf{X}}\right|_{\mathbf{X}_0} + O((\mathbf{X}-\mathbf{X}_0)^2) \tag{A.1}$$

Corollaries

To the dynamical system defined by (1) or (2) is associated a *tangent linear system* defined as follows:

$$\frac{d\delta\mathbf{X}}{dt} = J(\mathbf{X}_0)\delta\mathbf{X} \tag{A.2}$$

where

$$\delta\mathbf{X} = \mathbf{X}-\mathbf{X}_0, \quad \mathbf{X}_0 = \mathbf{X}(t_0) \quad \text{and}$$
$$\left.\frac{d\mathfrak{I}(\mathbf{X})}{d\mathbf{X}}\right|_{\mathbf{X}_0} = J(\mathbf{X}_0)$$

Corollary 1. *The nonlinear part condition implies the stability of the slow manifold. So, according to Proposition 3.3, the velocity varies slowly on the slow manifold. This involves that the functional jacobian $J(\mathbf{X}_0)$ varies slowly with time, i.e.*

$$\frac{dJ}{dt}(\mathbf{X}_0) = 0 \tag{A.3}$$

The solution of the tangent linear system (A.2) is written as:

$$\delta\mathbf{X} = e^{J(\mathbf{X}_0)(t-t_0)}\delta\mathbf{X}(t_0) \tag{A.4}$$

So,

$$\delta\mathbf{X} = \sum_{i=1}^{n} a_i \mathbf{Y}_{\lambda_i} \tag{A.5}$$

where n is the dimension of the eigenspace, a_i represents coefficients depending explicitly on the coordinates of space and implicitly on time and $\mathbf{Y}_{\lambda_i}$ the eigenvectors associated in the functional jacobian of the tangent linear system.

Corollary 2. *In the vicinity of the slow manifold the velocities of the dynamical system defined by (1) or (2) and that of the tangent linear system (4) merge.*

$$\frac{d\delta\mathbf{X}}{dt} = \mathbf{V}_T \approx \mathbf{V} \tag{A.6}$$

where $\mathbf{V}_T$ represents the velocity vector associated in the tangent linear system.

The *tangent linear system approximation* method consists in spreading the velocity vector field $\mathbf{V}$ on the eigenbasis associated to the eigenvalues of the functional jacobian of the *tangent linear system*.

Indeed, by taking account of (A.2) and (A.5) we have according to (A.6):

$$\begin{aligned}\frac{d\delta\mathbf{X}}{dt} &= J(\mathbf{X}_0)\delta\mathbf{X}\\ &= J(\mathbf{X}_0)\sum_{i=1}^{n} a_i\mathbf{Y}_{\lambda_i}\\ &= \sum_{i=1}^{n} a_i J(\mathbf{X}_0)\mathbf{Y}_{\lambda_i}\\ &= \sum_{i=1}^{n} a_i\lambda_i\mathbf{Y}_{\lambda_i}\end{aligned} \tag{A.7}$$

Thus, Corollary 2 provides:

$$\frac{d\delta\mathbf{X}}{dt} = \mathbf{V}_T \approx \mathbf{V} = \sum_{i=1}^{n} a_i\lambda_i\mathbf{Y}_{\lambda_i} \tag{A.8}$$

Equation (A.8) constitutes in dimension two (resp. dimension three) a condition called *collinearity* (resp. *coplanarity*) condition which provides the analytical equation of the *slow manifold* of a dynamical system defined by (1) or (2).

An alternative proposed by Rossetto *et al.* [1998] uses the “fast” eigenvector on the left associated to the “fast” eigenvalue of the transposed functional jacobian of the *tangent linear system*.

In this case the velocity vector field $\mathbf{V}$ is then orthogonal with the “fast” eigenvector on the left. This constitutes a condition called *orthogonality* condition which provides the analytical equation of the *slow manifold* of a dynamical system defined by (1) or (2).

These two conditions will be the subject of a detailed presentation in the following sections. Thereafter it will be supposed that the assumptions (H1) and (H2) are always checked.

Collinearity/coplanarity condition

Slow manifold equation of a two-dimensional dynamical system

Let us consider a dynamical system defined under the same conditions as (1) or (2). The eigenvectors associated to the eigenvalues of the functional jacobian of the *tangent linear system* are written as:

$$\mathbf{Y}_{\lambda_i}\begin{pmatrix}\lambda_i - \dfrac{\partial g}{\partial y}\\[2ex] \dfrac{\partial g}{\partial x}\end{pmatrix} \tag{A.9}$$

with

$$i = 1, 2$$

The projection of the velocity vector field $\mathbf{V}$ on the eigenbasis is written according to Corollary 2:

$$\begin{aligned}\frac{d\delta\mathbf{X}}{dt} = \mathbf{V}_T \approx \mathbf{V} &= \sum_{i=1}^{n} a_i\lambda_i\mathbf{Y}_{\lambda_i}\\ &= \alpha\mathbf{Y}_{\lambda_1} + \beta\mathbf{Y}_{\lambda_2}\end{aligned}$$

where α and β represent coefficients depending explicitly on the coordinates of space and implicitly on time and where $\mathbf{Y}_{\lambda_1}$ represents the “fast” eigenvector and $\mathbf{Y}_{\lambda_2}$ the “slow” eigenvector. The existence of an evanescent mode in the vicinity of the *slow manifold* implies according to Tihonv’s theorem [1952]: $\alpha \ll 1$. We deduce:

Proposition A.1. *A necessary and sufficient condition of obtaining the slow manifold equation of a two-dimensional dynamical system is that its velocity vector field* $\mathbf{V}$ *is collinear to the slow eigenvector* $\mathbf{Y}_{\lambda_2}$ *associated to the slow eigenvalue* λ_2 *of the functional jacobian of the tangent linear system. That is to say*:

$$\mathbf{V} \approx \beta\mathbf{Y}_{\lambda_2} \tag{A.10}$$

While using this collinearity condition, the equation constituting the first-order approximation in ε of the *slow manifold* of a two-dimensional dynamical system is written as:

$$\begin{gathered}\mathbf{V} \wedge \mathbf{Y}_{\boldsymbol{\lambda_2}} = \mathbf{0}\\ \Leftrightarrow\\ \left(\frac{\partial g}{\partial x}\right)\left(\frac{dx}{dt}\right) - \left(\lambda_2 - \frac{\partial g}{\partial y}\right)\left(\frac{dy}{dt}\right) = 0\end{gathered} \tag{A.11}$$

Slow manifold equation of a three-dimensional dynamical system

Let us consider a dynamical system defined under the same conditions as (1) or (2). The eigenvectors associated to the eigenvalues of the functional jacobian of the *tangent linear system* are written as:

$$\mathbf{Y}_{\lambda_i}\begin{pmatrix} \frac{1}{\varepsilon}\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}+\frac{1}{\varepsilon}\frac{\partial f}{\partial z}\left(\lambda_i-\frac{\partial g}{\partial y}\right) \\ \frac{1}{\varepsilon}\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}+\frac{\partial g}{\partial z}\left(\lambda_i-\frac{1}{\varepsilon}\frac{\partial f}{\partial x}\right) \\ -\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}+\left(\lambda_i-\frac{1}{\varepsilon}\frac{\partial f}{\partial x}\right)\left(\lambda_i-\frac{\partial g}{\partial y}\right) \end{pmatrix} \tag{A.12}$$

with

$$i = 1, 2, 3$$

The projection of the velocity vector field $\mathbf{V}$ on the eigenbasis is written according to Corollary 2:

$$\frac{d\delta\mathbf{X}}{dt} = \mathbf{V}_T \approx \mathbf{V} = \sum_{i=1}^{n} a_i\lambda_i\mathbf{Y}_{\lambda_i}$$
$$= \alpha\mathbf{Y}_{\lambda_1} + \beta\mathbf{Y}_{\lambda_2} + \delta\mathbf{Y}_{\lambda_3}$$

where α, β and δ represent coefficients depending explicitly on the coordinates of space and implicitly on time and where $\mathbf{Y}_{\lambda_1}$ represents the "fast" eigenvector and $\mathbf{Y}_{\lambda_2}$, $\mathbf{Y}_{\lambda_3}$ the "slow" eigenvectors. The existence of an evanescent mode in the vicinity of the *slow manifold* implies according to Tihonv's theorem [1952]: $\alpha \ll 1$. We deduce:

Proposition A.2. *A necessary and sufficient condition of obtaining the slow manifold equation of a three-dimensional dynamical system is that its velocity vector field* $\mathbf{V}$ *is coplanar to the slow eigenvectors* $\mathbf{Y}_{\lambda_2}$ *and* $\mathbf{Y}_{\lambda_3}$ *associated to the slow eigenvalues* λ_2 *and* λ_3 *of the functional jacobian of the tangent linear system. That is to say*:

$$\mathbf{V} \approx \beta\mathbf{Y}_{\lambda_2} + \delta\mathbf{Y}_{\lambda_3} \tag{A.13}$$

While using this coplanarity condition, the equation constituting the first-order approximation in ε of the *slow manifold* of a three-dimensional dynamical system is written as:

$$\det(\mathbf{V}, \mathbf{Y}_{\lambda_2}, \mathbf{Y}_{\lambda_3}) = 0 \Leftrightarrow \mathbf{V}\cdot(\mathbf{Y}_{\lambda_2}\wedge\mathbf{Y}_{\lambda_3}) = 0 \tag{A.14}$$

Orthogonality condition

Slow manifold equation of a two-dimensional dynamical system

Let us consider a dynamical system defined under the same conditions as (1) or (2). The eigenvectors associated to the eigenvalues of the transposed functional jacobian of the *tangent linear system* are written as:

$${}^t\mathbf{Y}_{\lambda_i}\begin{pmatrix} \lambda_i-\frac{\partial g}{\partial y} \\ \frac{1}{\varepsilon}\frac{\partial f}{\partial y} \end{pmatrix} \tag{A.15}$$

with

$$i = 1, 2$$

${}^t\mathbf{Y}_{\lambda_1}$ represents the "fast" eigenvector on the left associated to the dominant eigenvalue, i.e. the largest eigenvalue in absolute value and ${}^t\mathbf{Y}_{\lambda_2}$ is the "slow" eigenvector on the left.

But since according to Rossetto *et al.* [1998], the velocity vector field $\mathbf{V}$ is perpendicular to the "fast" eigenvector on the left ${}^t\mathbf{Y}_{\lambda_1}$, we deduce:

Proposition A.3. *A necessary and sufficient condition of obtaining the slow manifold equation of a two-dimensional dynamical system is that its velocity vector field* $\mathbf{V}$ *is perpendicular to the fast eigenvector* ${}^t\mathbf{Y}_{\lambda_1}$ *on the left associated to the fast eigenvalue* λ_1 *of the transposed functional jacobian of the tangent linear system. That is to say*:

$$\mathbf{V}\perp{}^t\mathbf{Y}_{\lambda_1} \tag{A.16}$$

While using this orthogonality condition, the equation constituting the first-order approximation in ε of the *slow manifold* of a two-dimensional dynamical system is written as:

$$\mathbf{V}\cdot{}^t\mathbf{Y}_{\lambda_1} = 0$$
$$\Leftrightarrow$$
$$\left(\lambda_1-\frac{\partial g}{\partial y}\right)\left(\frac{dx}{dt}\right)+\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right) = 0 \tag{A.17}$$

Slow manifold equation of a three-dimensional dynamical system

Let us consider a dynamical system defined under the same conditions as (1) or (2). The eigenvectors associated to the eigenvalues of the transposed functional jacobian of the *tangent linear system* are written as:

$${}^t\mathbf{Y}_{\lambda_i}\begin{pmatrix} \frac{\partial g}{\partial x}\frac{\partial h}{\partial y}+\frac{\partial h}{\partial x}\left(\lambda_i-\frac{\partial g}{\partial y}\right) \\ \frac{1}{\varepsilon}\frac{\partial f}{\partial y}\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}\left(\lambda_i-\frac{1}{\varepsilon}\frac{\partial f}{\partial x}\right) \\ -\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}+\left(\lambda_i-\frac{1}{\varepsilon}\frac{\partial f}{\partial x}\right)\left(\lambda_i-\frac{\partial g}{\partial y}\right) \end{pmatrix} \tag{A.18}$$

with

$$i = 1, 2, 3$$

${}^t\mathbf{Y}_{\lambda_1}$ represents the "fast" eigenvector on the left associated to the dominant eigenvalue, i.e. the largest eigenvalue in absolute value and ${}^t\mathbf{Y}_{\lambda_2}$, ${}^t\mathbf{Y}_{\lambda_3}$ are the "slow" eigenvectors on the left.

But since according to Rossetto *et al.* [1998], the velocity vector field $\mathbf{V}$ is perpendicular to the "fast" eigenvector on the left ${}^t\mathbf{Y}_{\lambda_1}$, we deduce:

Proposition A.4. *A necessary and sufficient condition of obtaining the slow manifold equation of a three-dimensional dynamical system is that its velocity vector field* $\mathbf{V}$ *is perpendicular to the fast eigenvector* ${}^t\mathbf{Y}_{\lambda_1}$ *on the left associated in the fast eigenvalue* λ_1 *of the transposed functional jacobian of the tangent linear system. That is to say*:

$$\mathbf{V} \perp {}^t\mathbf{Y}_{\lambda_1} \tag{A.19}$$

While using this orthogonality condition, the equation constituting the first-order approximation in ε of the *slow manifold* of a three-dimensional dynamical system is written as:

$$\mathbf{V} \perp {}^t\mathbf{Y}_{\lambda_1} \Leftrightarrow \mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_1} = 0 \tag{A.20}$$

Equivalence of both conditions

Proposition A.5. *Both necessary and sufficient collinearity/coplanarity and orthogonality conditions providing the slow manifold equation are equivalent.*

Proof of Proposition 5 in Dimension Two. In dimension two, the *slow manifold* equation may be obtained while considering that the velocity vector field $\mathbf{V}$ is:

- either collinear to the "slow" eigenvector $\mathbf{Y}_{\lambda_2}$
- either perpendicular to the "fast" eigenvector on the left ${}^t\mathbf{Y}_{\lambda_1}$

There is equivalence between both conditions provided that the "fast" eigenvector on the left ${}^t\mathbf{Y}_{\lambda_1}$ is orthogonal to the "slow" eigenvector $\mathbf{Y}_{\lambda_2}$. Both co-ordinates of these eigenvectors defined in the above section make it possible to express their scalar product:

$$\begin{aligned} {}^t\mathbf{Y}_{\lambda_1} \cdot \mathbf{Y}_{\lambda_2} &= \lambda_1\lambda_2 - \frac{\partial g}{\partial y}(\lambda_1 + \lambda_2) \\ &\quad + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right) \end{aligned}$$

While using the trace and determinant of the functional jacobian of the *tangent linear system*, we have:

$$\begin{aligned} {}^t\mathbf{Y}_{\lambda_1} \cdot \mathbf{Y}_{\lambda_2} &= \frac{1}{\varepsilon}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) \\ &\quad - \left(\frac{\partial g}{\partial y}\right)\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) \\ &\quad + \left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right) \\ &= 0 \end{aligned}$$

So,

$${}^t\mathbf{Y}_{\lambda_1} \perp \mathbf{Y}_{\lambda_2} \tag{A.21}$$

Thus, collinearity and orthogonality conditions are completely equivalent. ■

Proof of Proposition 5 in Dimension Three. In dimension three, the *slow manifold* equation may be obtained while considering that the velocity vector field $\mathbf{V}$ is:

- either coplanar to the "slow" eigenvectors $\mathbf{Y}_{\lambda_2}$ and $\mathbf{Y}_{\lambda_3}$
- either perpendicular to the "fast" eigenvector on the left ${}^t\mathbf{Y}_{\lambda_1}$

There is equivalence between both conditions provided that the "fast" eigenvector on the left ${}^t\mathbf{Y}_{\lambda_1}$ is orthogonal to the plane containing the "slow" eigenvectors $\mathbf{Y}_{\lambda_2}$ and $\mathbf{Y}_{\lambda_3}$. While using the coordinates of these eigenvectors defined in the above section, the sum and the product (also the square of the sum and the product) of the eigenvalues of the functional jacobian of the *tangent linear system*, the following equality is demonstrated:

$$\mathbf{Y}_{\lambda_2} \wedge \mathbf{Y}_{\lambda_3} = {}^t\mathbf{Y}_{\lambda_1} \tag{A.22}$$

Thus, coplanarity and orthogonality conditions are completely equivalent. ■

Note: In dimension three, numerical studies shown significant differences in the plot of *slow manifold* according to whether one or the other of both conditions (coplanarity (A.14) or orthogonality (A.20)) were used. These differences come from the fact that each of these two conditions uses one or two eigenvectors whose coordinates are expressed according to the eigenvalues of the functional jacobian of the *tangent linear system*. Since these eigenvalues can be complex or real according to their localization in the phase space the plot of the analytical equation of the *slow manifold* can be difficult even impossible. Also to solve this problem it is necessary to make the analytical equation of the *slow manifold* independent of the eigenvalues. This can be carried out by multiplying each equation of the *slow manifold* by one or two "conjugated" equations. The equation obtained will be presented in each case (dimension two and three) in the next section.

Slow manifold equation independent of the eigenvectors

Proposition A.6. *Slow manifold equations of a dynamical system obtained by the collinearity/coplanarity and orthogonality conditions are equivalent.*

Proof of Proposition 6 in Dimension Two. In order to demonstrate the equivalence between the *slow manifold* equations obtained by each condition, they should be expressed independently of the eigenvalues. So let us multiply each equation (A.11) then (A.17) by its "conjugated" equation, i.e. an equation in which the eigenvalue λ_1 (resp. λ_2) is replaced by the eigenvalue λ_2 (resp. λ_1). Let us take note that the "conjugated" equation (A.11) corresponds to the collinearity condition between the velocity vector field $\mathbf{V}$ and the eigenvector $\mathbf{Y}_{\lambda_1}$. The product of Eq. (A.11) by its "conjugated" equation is written as:

$$(\mathbf{V} \wedge \mathbf{Y}_{\lambda_1}) \cdot (\mathbf{V} \wedge \mathbf{Y}_{\lambda_2}) = 0$$

So, while using the trace and the determinant of the functional jacobian of the *tangent linear system* we have:

$$\left(\frac{\partial g}{\partial x}\right)\left[\left(\frac{\partial g}{\partial x}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)\left(\frac{dx}{dt}\right) \times \left(\frac{dy}{dt}\right) - \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right)^2\right] = 0 \tag{A.23}$$

In the same manner, the product of Eq. (A.17) by its "conjugated" equation which corresponds to the orthogonality condition between velocity vector field $\mathbf{V}$ and the eigenvector ${}^t\mathbf{Y}_{\lambda_2}$ is written as:

$$(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_1})(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_2}) = 0$$

So, while using the trace and the determinant of the functional jacobian of the *tangent linear system* we have:

$$\left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)\left[\left(\frac{\partial g}{\partial x}\right)\left(\frac{dx}{dt}\right)^2 - \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}\right)\left(\frac{dx}{dt}\right) \times \left(\frac{dy}{dt}\right) - \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right)^2\right] = 0 \tag{A.24}$$

Both Eqs. (A.23) and (A.24) are equal provided that:

$$\left(\frac{\partial g}{\partial x}\right) \neq 0 \quad \text{and} \quad \left(\frac{1}{\varepsilon}\frac{\partial f}{\partial y}\right) \neq 0$$

These two last conditions are, according to the definition of a dynamical system, satisfied because if they were not both differential equations that make the system completely uncoupled, it would not be a system anymore. Thus, the equations obtained by the collinearity and orthogonality conditions are equivalent.

$$(\mathbf{V} \wedge \mathbf{Y}_{\lambda_1}) \cdot (\mathbf{V} \wedge \mathbf{Y}_{\lambda_2}) = 0$$
$$\Leftrightarrow$$
$$(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_1})(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_2}) = 0 \tag{A.25}$$

■

Equations (A.23) and (A.24) provide the *slow manifold* equation of a two dimensional dynamical system independently of the eigenvalues of the functional jacobian of the *tangent linear system*. In order to express them, we adopt the following notations for:

- the velocity vector field

$$\mathbf{V}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$$

- the functional jacobian

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- the eigenvectors coordinates (A.9)

$$\mathbf{Y}_{\lambda_i}\begin{pmatrix}\lambda_i - d\\ c\end{pmatrix}$$

with

$$i = 1, 2$$

Equations (A.23) and (A.24) providing the *slow manifold* equation of a two-dimensional dynamical system independently of the eigenvalues of the functional jacobian of the *tangent linear system* are then written as:

$$c\dot{x}^2 - (a - d)\,\dot{x}\dot{y} - b\dot{y}^2 = 0 \tag{A.26}$$

$$\phi = \sum_{i,j=0}^{2} \alpha_{ij}\dot{x}^i\dot{y}^j = 0 \quad \text{with}$$

$$\alpha_{ij} = \begin{cases} = 0 \text{ si } i + j \neq 2 \\ \neq 0 \text{ si } i + j = 2 \end{cases} \tag{A.27}$$

■

Proof of Proposition 6 in Dimension Three. In order to demonstrate the equivalence between the *slow manifold* equations obtained by each condition, the same step as that exposed in the above section is applied. The *slow manifold* equation should be expressed independently of the eigenvalues. In dimension three, each equation (A.14) then (A.20) must be multiplied by two "conjugated" equations obtained by circular shifts of the eigenvalues. Let us take note that the first of the "conjugated" equations (A.14) corresponds to the coplanarity condition between the velocity vector field $\mathbf{V}$ and the eigenvectors $\mathbf{Y}_{\lambda_1}$ and $\mathbf{Y}_{\lambda_2}$, the second corresponds to the coplanarity condition between the velocity vector field $\mathbf{V}$ and the eigenvectors $\mathbf{Y}_{\lambda_1}$ and $\mathbf{Y}_{\lambda_3}$. The product of Eq. (A.14) by its "conjugated" equation is written as:

$$\begin{aligned} &[\mathbf{V} \cdot (\mathbf{Y}_{\lambda_1} \wedge \mathbf{Y}_{\lambda_2})] \cdot [\mathbf{V} \cdot (\mathbf{Y}_{\lambda_2} \wedge \mathbf{Y}_{\lambda_3})] \\ &\quad \cdot [\mathbf{V} \cdot (\mathbf{Y}_{\lambda_1} \wedge \mathbf{Y}_{\lambda_3})] = 0 \end{aligned} \tag{A.28}$$

In the same manner, the product of Eq. (A.20) by its "conjugated" equation which corresponds to the orthogonality condition between the velocity vector field $\mathbf{V}$ and the eigenvector ${}^t\mathbf{Y}_{\lambda_2}$ and, the orthogonality condition between the velocity vector field $\mathbf{V}$ and the eigenvector ${}^t\mathbf{Y}_{\lambda_3}$ is written as:

$$(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_1})(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_2})(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_3}) = 0 \tag{A.29}$$

By using Eq. (A.22) and all the circular shifts which result from this we demonstrate that Eqs. (A.28) and (A.29) are equal. Thus, the equations obtained by the coplanarity and orthogonality conditions are equivalent.

$$\begin{aligned} &[\mathbf{V} \cdot (\mathbf{Y}_{\lambda_1} \wedge \mathbf{Y}_{\lambda_2})] \cdot [\mathbf{V} \cdot (\mathbf{Y}_{\lambda_2} \wedge \mathbf{Y}_{\lambda_3})] \\ &\quad \cdot [\mathbf{V} \cdot (\mathbf{Y}_{\lambda_1} \wedge \mathbf{Y}_{\lambda_3})] = 0 \\ &\qquad\qquad \Leftrightarrow \\ &(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_1})(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_2})(\mathbf{V} \cdot {}^t\mathbf{Y}_{\lambda_3}) = 0 \end{aligned} \tag{A.30}$$

■

Equations (A.28) and (A.29) provide the *slow manifold* equation of a three-dimensional dynamical system independently of the eigenvalues of the functional jacobian of the *tangent linear system*. In order to express them, we adopt the following notations for:

- the velocity vector field

$$\mathbf{V}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

- the functional jacobian

$$J = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

- the "slow" eigenvectors coordinates (A.12)

$$\mathbf{Y}_{\lambda_i}\begin{pmatrix} bf + c(\lambda_i - e) \\ cd + f(\lambda_i - a) \\ -bd + (\lambda_i - a)(\lambda_i - e) \end{pmatrix}$$

  with

$$i = 2, 3$$

- the "fast" eigenvectors coordinates on the left (A.18)

$${}^t\mathbf{Y}_{\lambda_1}\begin{pmatrix} hd + g(\lambda_1 - e) \\ bg + h(\lambda_1 - a) \\ -bd + (\lambda_1 - a)(\lambda_1 - e) \end{pmatrix}$$

Starting from the coplanarity condition (A.14) and while replacing the eigenvectors by their coordinates (A.12) and while removing all the eigenvalues λ_2 and λ_3 thanks to the sum and the products (and also the square of the sum and the products) of the eigenvalues of the functional jacobian of the *tangent linear system*, we obtain the following equation:

$$A_1\dot{x} - B_1\dot{y} + C\dot{z} = 0 \tag{A.31}$$

with

$$\begin{aligned} A_1 &= f\lambda_1^2 - (ef + if + cd)\lambda_1 + efi \\ &\quad + cdi - cfg - f^2h \\ B_1 &= c\lambda_1^2 - (ac + ic + bf)\lambda_1 + aci \\ &\quad + bfi - c^2g - cfh \\ C &= bf^2 - c^2d + cf(a - e) \end{aligned} \tag{A.32}$$

Equation (A.31) is absolutely identical to that one would obtain by the orthogonality condition (A.20). Let us multiply Eq. (A.31) by its "conjugated" equations in λ_2 and λ_3, i.e. by $(A_2\dot{x} - B_2\dot{y} + C\dot{z})$ and $(A_3\dot{x} - B_3\dot{y} + C\dot{z})$. The coefficients A_i, B_i are obtained by replacing in Eq. (A.32) the eigenvalue λ_1 by eigenvalue λ_2

Table 2. Slow manifold analytical equation of a two-dimensional dynamical system.

$\phi = \sum_{i,j=0}^{2} \alpha_{ij} \dot{x}^i \dot{y}^j = 0$ with $\alpha_{ij} = \begin{cases} = 0 \text{ si } i+j \neq 2 \\ \neq 0 \text{ si } i+j = 2 \end{cases}$
$\alpha_{20} = c$
$\alpha_{11} = -(a-d)$
$\alpha_{02} = -b$

Table 3. Slow manifold analytical equation of a three-dimensional dynamical system.

$\phi = \sum_{i,j,k=0}^{3} \alpha_{ijk} \dot{x}^i \dot{y}^j \dot{z}^k = 0$ with $\alpha_{ijk} = \begin{cases} = 0 \text{ si } i+j+k \neq 3 \\ \neq 0 \text{ si } i+j+k = 3 \end{cases}$
$\alpha_{300} = d^2h + dgi - fg^2 - dge$
$\alpha_{030} = ch^2 + abh - b^2g - ibh$
$\alpha_{003} = c^2d + cfe - bf^2 - cfa$
$\alpha_{210} = bdg + aeg - e^2g + cg^2 - 2adh - 2fgh + deh - agi + egi + dhi$
$\alpha_{120} = -abg + 2beg + a^2h - fh^2 - bdh - aeh + 2cgh - bgi - ahi + ehi$
$\alpha_{201} = -bd^2 + ade - cdg + 2afg + 2dfh - efg - adi - dei - fgi + di^2$
$\alpha_{102} = acd + cde - a^2f - 2bdf + aef + cfg + f^2h - 2cdi + afi - efi$
$\alpha_{021} = b^2d - abe - 2bcg - 2ceh + ach + bfh + abi + bei + chi - bi^2$
$\alpha_{012} = 2bcd - ace + ce^2 - abf - bef - c^2g - cfh + aci - cei + 2bfi$
$\alpha_{111} = abd - a^2e - bde + ae^2 - acg + 3bfg - 3cdh + efh + a^2i - e^2i + cgi - fhi - ai^2 + ei^2$

then by eigenvalue λ_3 respectively for $i = 2, 3$. We obtain:

$$(A_1\dot{x} - B_1\dot{y} + C\dot{z})(A_2\dot{x} - B_2\dot{y} + C\dot{z}) \times (A_3\dot{x} - B_3\dot{y} + C\dot{z}) = 0 \tag{A.33}$$

So,

$$\phi = \sum_{i,j,k=0}^{3} \alpha_{ijk} \dot{x}^i \dot{y}^j \dot{z}^k = 0 \quad \text{with} \quad \alpha_{ijk} = \begin{cases} = 0 \text{ si } i+j+k \neq 3 \\ \neq 0 \text{ si } i+j+k = 3 \end{cases} \tag{A.34}$$

By developing this expression we obtain a polynomial comprising terms of the sum and product of eigenvalues and also of the square of sum and product of eigenvalues and which are directly connected to the elements of the functional jacobian matrix of the *tangent linear system*. The equation obtained is the result of a demonstration (available by request to the authors) which establishes a relation between the coefficients of this polynomial and the elements of the functional jacobian matrix of the *tangent linear system*.

The expression (A.34) represents the *slow manifold* equation of a three-dimensional dynamical system independently of the eigenvalues of the functional jacobian of the *tangent linear system*. Both expressions (A.27) and (A.34) are also available at the address: http://ginoux.univ-tln.fr.