Dynamical Systems Analysis Using Differential Geometry

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Abstract

This paper aims to analyze trajectories behavior and attractor structure of chaotic dynamical systems with the Differential Geometry and Mechanics formalism. Applied to slow-fast autonomous dynamical systems (S-FADS), this approach provides: on the one hand a kinematics interpretation of the trajectories motion, and on the other hand, a direct determination of the slow manifold equation. The attractivity of this manifold established with a new criterion makes it possible to ensure attractors stability. Then, a qualitative description of the geometrical structure of the attractor is presented. It consists in considering it as the deployment in the space phase of a special submanifold that is called singular manifold. The attractor can be obtained by integration of initial conditions taken on this singular manifold. Applications of this method are made for the following models: cubic-Chua, and Volterra-Gause.

Introduction

In the *Mechanics* formalism the solution of a dynamical system is considered as the co-ordinates of a moving point M at the instant *t*. Then, three kinematics variables are attached to this point which represents the “trajectory curve”:

$\vec{X}(t)$: parametric representation of chaotic orbit,

$\vec{V}(t)$: instantaneous velocity vector,

$\vec{\gamma}(t)$: instantaneous acceleration vector.

The Differential Geometry allows to use the Frénet frame [2] which is moving with the “trajectory curve” and directed towards its motion, consists in, a unit tangent vector to the “trajectory curve”, a unit normal vector, directed towards the interior of the concavity of the curve and a unit binormal vector to the trajectory curve so that the trihedron $\left(\vec{\tau}, \vec{\beta}, \vec{\nu}\right)$ is direct (Cf. Fig.1.).

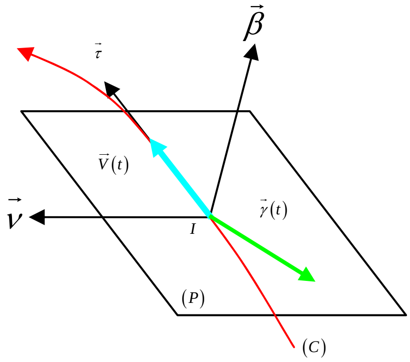

Fig. 1. Frénet frame and osculating plane.

In this moving frame the instantaneous acceleration vector may be decomposed in a tangential and normal component both depending on instantaneous velocity and acceleration vectors directions.

$$\gamma_\tau = \frac{\vec{\gamma}.\vec{V}}{\left\|\vec{V}\right\|} \qquad \gamma_\nu = \frac{\left\|\vec{\gamma} \wedge \vec{V}\right\|}{\left\|\vec{V}\right\|} \qquad (1)$$

The osculating plane [7] to the "trajectory curve" presented in Fig. 1. is the plane passing through a fixed point I and spanned by the instantaneous velocity and acceleration vectors. Its equation may be provided by the coplanarity condition (2).

$$\forall\ M \in\ (P)\ \Leftrightarrow\ \exists\ (\mu, \eta) \in \mathbb{R}^2\ /\ \overrightarrow{IM} = \mu\vec{V} + \eta\vec{\gamma}$$

This coplanarity condition may be written:

$$\overrightarrow{IM}\ .\left(\vec{V} \wedge \vec{\gamma}\right) = 0 \qquad (2)$$

In this formalism, trajectory presents two "metric properties":

- curvature which expresses the rate of change of the tangent when moving along the "trajectory curve". $\Re$ represents the radius of curvature.
- torsion which measures, roughly speaking, the magnitude and sense of deviation of the "trajectory curve" from the osculating plane, or, in other words, the rate of change of the osculating plane. $\Im$ represents the radius of torsion.

$$\frac{1}{\Re} = \frac{\left\| \vec{\gamma} \wedge \vec{V} \right\|}{\left\| \vec{V} \right\|^3} = \frac{\gamma_v}{\left\| \vec{V} \right\|^2}$$
$$\frac{1}{\Im} = -\frac{\dot{\vec{\gamma}} \cdot \left(\vec{\gamma} \wedge \vec{V} \right)}{\left\| \vec{\gamma} \wedge \vec{V} \right\|^2} \qquad (3)$$

Then, the use of the instantaneous acceleration vector makes it possible to delimit the *slow* and *fast* domains of the phase space,

Definition

The domain of the phase space in which the tangential component of the instantaneous acceleration vector is negative, i.e., the domain in which the system is decelerating is called *slow* domain. The domain of the phase space in which the tangential component of the instantaneous acceleration vector is positive, i.e., the domain in which the system is accelerating is called *fast* domain.

New Method of Determination of the Slow Manifold Equation

Applying both formalisms, recalled in the previous section, to slow-fast autonomous dynamical systems (S-FADS) or to autonomous dynamical systems which can be considered as slow-fast (CAS-FADS), i.e., systems whose functional jacobian matrix has a "fast eigenvalue" which is a real, negative and dominant on a large domain of the phase space [6], a new method of determination of the slow manifold equation is proposed:

Proposition 1

The equation of the osculating plane (P) passing through a fixed point I of a dynamical system (S-FADS or CAS-SFADS) and spanned by the instantaneous, velocity vector $\vec{V}$ and acceleration vector $\vec{\gamma}$, is the slow manifold equation associated to this system.

In order to specify the attractivity of this manifold a new criterion based on the envelope theory is also proposed [4]:

Proposition 2

The attractivity of the slow manifold is given by the sign of the torsion which constitutes the envelope of the slow manifold defined by the osculating plane.

It can be shown [4] that the total differential with respect to time of the osculating plane equation defined by the coplanarity condition (2) corresponds to the torsion. Thus, the location of the points where the torsion vanishes corresponds to the location of the points where the osculating plane is stationary.

Then, a qualitative description of the attractor structure is presented with the introduction of a submanifold called singular manifold:

Proposition 3

The singular manifold is defined like the location of the points belonging to the slow manifold and for which the tangential component of the instantaneous acceleration vector vanishes. This leads to the following equations:

$$\begin{cases} \phi = 0 \\ \gamma_\tau = 0 \end{cases} \quad (4)$$

This one-dimensional manifold is a submanifold of the slow manifold.

Let us consider the location of the points obtained by integration in a given time of initial conditions taken on this manifold. Each point being the iterated to the antecedent point. They constitute a submanifold which also belongs to the attractor. The whole of these manifolds corresponds to different points of integration making it possible to reconstitute the attractor by redeployment of the singular manifold.

Applications and Numerical Simulations

Applications of this new method are made for the following models: cubic-Chua and Volterra-Gause.

Cubic-Chua's circuit

Let's first recall the cubic Chua's circuit [1] which is a (S-FADS). Parameters used are:

$$\varepsilon = 0.05,\ \mu = 2$$

$$\vec{V} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon}\left(z - \frac{44}{3}x^3 - \frac{41}{2}x^2 - \mu x \right) \\ -z \\ -0.7x + y + 0.24z \end{pmatrix} \quad (5)$$

In the Fig.2. is plotted the slow manifold equation associated to the cubic-Chua's circuit.

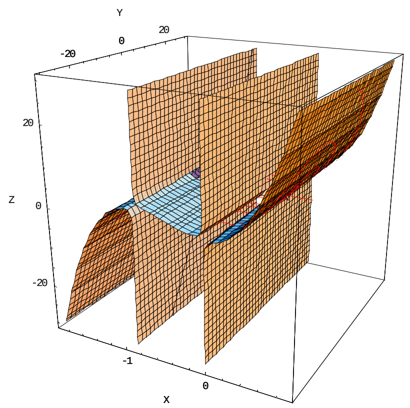

Fig. 2. Slow manifold equation associated to the cubic-Chua's circuit defined by the osculating plane method

In the Fig.3. is plotted the location of the point where the torsion associated to the cubic-Chua's circuit vanishes.

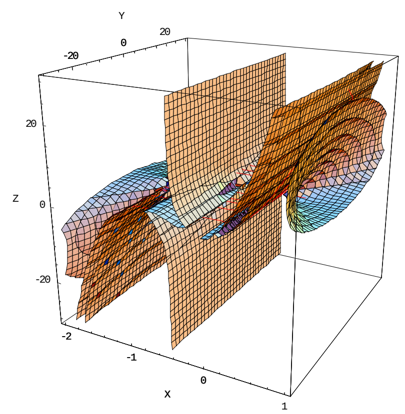

Fig. 3. Location of the point where the torsion associated to the cubic-Chua's circuit vanishes, i.e., where the osculating plane is stationary.

Volterra-Gause model

In order to illustrate the concept of deployment let's apply it on a three-dimensional predator-prey model elaborated by Ginoux et al. [3].This model consisted of a prey, a predator and top-predator has been named Volterra-Gause because it combines the original model of V. Volterra (1926) incorporating a logisitic limitation of P.F. Verhulst (1838) type on the growth of the prey and a limitation of G.F. Gause (1935) type on the intensity of the predation of the predator on the prey and of top-predator on the predator.

$$\vec{V} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \vec{\Im} \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \end{pmatrix} = \begin{pmatrix} \frac{1}{\xi}\left(x(1-x) - x^{\frac{1}{2}} y \right) \\ -\delta_1 y + x^{\frac{1}{2}} y - y^{\frac{1}{2}} z \\ \varepsilon z \left(y^{\frac{1}{2}} - \delta_2 \right) \end{pmatrix} \quad (6)$$

Parameters used are:

$$\xi = 0.866,\ \varepsilon = 1.428,\ \delta_1 = 0.577,\ \delta_2 = 0.376$$

This model exhibits a chaotic attractor in the snail shell shape presented in Fig. 4. The use of the algorithm developed by Wolf et al. [8] made it possible to compute what can be regarded as its Lyapunov exponents: (+0.035, 0.000, −0.628).

Then, the Kaplan-Yorke [5] conjecture provided the following Lyapunov dimension: 2.06. So, the fractal dimension of this chaotic attractor is close to that of a surface and it is thus possible to consider a deployment of a singular manifold. Taking some points on the slow manifold for which the tangential component of the instantaneous acceleration vector vanishes, and joining these points, a “line” or more generally, a “curve” is formed. Then, using numerical integration, this “curve” (resp. “line”) is deployed through the phase space and its deployment reconstitutes to the attractor shape. The result is plotted in Fig. 4.

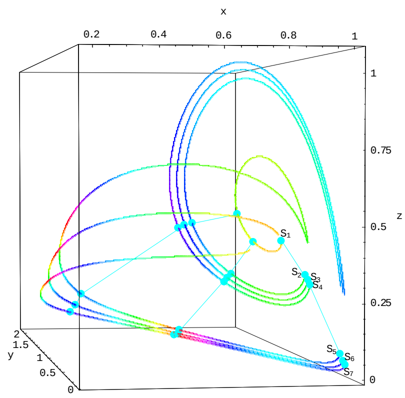

Fig. 4. Deployment of the singular manifolds (S_1, S_2) joining the singular points J and K of the system (6)

Conclusions and Discussions

The use of *Mechanics* and *Differential Geometry* formalism provided on one hand, a kinematics interpretation of the nature of the motion of chaotic trajectories, and on the other hand, a direct determination of the slow manifold equation associated to (S-FADS) or to (CAS-FADS). It is obvious that on the slow manifold, provided by the osculating plane method, the “trajectory curve” is decelerating.

Moreover, the introduction of the singular manifold which can reconstitute the attractor by successive integrations of points taken on this submanifold, i.e., by redeployment, provides a qualitative description of its structure.

The *Mechanics* formalism and more precisely the radius of curvature and the torsion could be useful to go further in the geometrical description and thus in the understanding of the attractor structure.

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