DIFFERENTIAL GEOMETRY APPLICATIONS TO NONLINEAR OSCILLATORS ANALYSIS

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Abstract—*In this work a new approach to stability analysis is applied to nonlinear oscillators. Based on the use of local metrics properties of curvature and torsion resulting from Differential Geometry and while considering trajectory curves as plane or space curves, these properties directly provide their slow manifold analytical equation and its stable and unstable parts. Van der Pol and Colpitts models emphasize the application of this method to electronic systems.*

I. Introduction

Stability analysis of nonlinear and chaotic dynamical systems lead to the research of invariant sets such as invariant manifolds [6]. During the second part of the last century various methods have been developed in order to determine the *slow invariant manifold* analytical equation associated with *nonlinear and chaotic dynamical systems*. The seminal works of Tikhonov [15], Wasow [17], Cole [1], O'Malley [9] and Fenichel [3] to name but a few, gave rise to the so-called *Geometric Singular Perturbation theory* and the problem for finding the *slow invariant manifold* analytical equation turned into a regular perturbation problem. In this work an alternative approach [4] to *singular perturbation methods* is applied to nonlinear oscillators. The attractivity or repulsivity of such *slow invariant manifold* is characterized while using a criterion proposed by Henri Poincaré [10]-[13].

II. Slow Manifold of Dynamical Systems

A. Dynamical system

In the following we consider a system of differential equations defined in a compact E included in $\mathbb{R}$:

$$\frac{d\vec{X}}{dt} = \overrightarrow{\Im}\left(\vec{X}\right) \tag{1}$$

with

$$\vec{X} = [x_1, x_2, ..., x_n]^t \in E \subset \mathbb{R}^{\ltimes}$$

and

$$\overrightarrow{\Im}\left(\vec{X}\right) = \left[f_1\left(\vec{X}\right), f_2\left(\vec{X}\right), ..., f_n\left(\vec{X}\right)\right]^t \in E \subset \mathbb{R}^{\ltimes}$$

The vector $\overrightarrow{\Im}\left(\vec{X}\right)$ defines a velocity vector field in E whose components f_i which are supposed to be continuous and infinitely differentiable with respect to all x_i and t, i.e., are C^{∞} functions in E and with values included in $\mathbb{R}$, satisfy the assumptions of the Cauchy-Lipschitz theorem [2]. A solution of this system is an integral curve $\vec{X}(t)$ tangent to $\overrightarrow{\Im}$ whose values define the *states* of the *dynamical system* described by the Eq. (1). Since none of the components f_i of the velocity vector field depends here explicitly on time, the system is said to be *autonomous*.

In this approach trajectory curves integral of nonlinear and chaotic dynamical systems are considered as plane or space curve. The framework of *Mechanics* will provide an interpretation of the behavior of the *trajectory curves* during the various phases of their motion in terms of *kinematics* variables: velocity and acceleration.

B. Kinematics vector functions

The integral of system (1) may be associated with the co-ordinates, i.e., with the position, of a point M at the instant t. This integral curve defined by the vector function $\vec{X}(t)$ of the scalar variable t represents the *trajectory* of M.

B.1 Instantaneous velocity vector

As the vector function $\vec{X}(t)$ of the scalar variable t represents the *trajectory* of M, the total derivative of $\vec{X}(t)$ is the vector function $\overrightarrow{V}(t)$ of the scalar variable t which represents the instantaneous velocity vector of the mobile M at the instant t; namely:

$$\overrightarrow{V}(t) = \frac{d\vec{X}}{dt} = \overrightarrow{\Im}\left(\vec{X}\right) \tag{2}$$

The instantaneous velocity vector $\overrightarrow{V}(t)$ is supported by the tangent to the *trajectory*.

B.2 Instantaneous acceleration vector

As the instantaneous vector function $\overrightarrow{V}(t)$ of the scalar variable t represents the velocity vector of M, the total derivative of $\overrightarrow{V}(t)$ is the vector function $\overrightarrow{\gamma}(t)$ of the scalar variable t which represents the instantaneous acceleration vector of the mobile M at the instant t; namely:

$$\overrightarrow{\gamma}(t) = \frac{d\overrightarrow{V}}{dt} \tag{3}$$

C. Trajectory curve properties

The framework of *Differential Geometry* will allow a study of the metric properties of the *trajectory curve*, i.e., *curvature* and *torsion* whose definitions are recalled in this section. A presentation of these concepts may be found in Struik [14], Kreyzig [7] or Gray [5].

C.1 Parametrization of the trajectory curve

The *trajectory curve* $\vec{X}(t)$ integral of the dynamical system defined by (1) or by (2), is described by the motion of a current point M position of which depends on a variable parameter: the time. This curve can also be defined by its parametric representation relative in a frame:

$$x_1 = F_1(t)\,,\ x_2 = F_2(t)\,,\ \ldots,\ x_n = F_n(t)$$

where the F_i functions are continuous, C^{∞} functions (or C^{r+1} according to the above assumptions) in E and with values in $\mathbb{R}$. Thus, the *trajectory curve* $\vec{X}(t)$ integral of the dynamical system defined by (1) or by (2), can be considered as a *plane curve* or as a *space curve* having certain metric properties like *curvature* and *torsion* which will be defined below.

C.2 Curvature of the trajectory curve

Let's consider the *trajectory curve* $\vec{\chi}(t)$ having in M an instantaneous velocity vector $\overrightarrow{V}(t)$ and an instantaneous acceleration vector $\overrightarrow{\gamma}(t)$, the *curvature*, which expresses the rate of changes of the tangent to the *trajectory curve*, is defined by:

$$\frac{1}{\Re} = \frac{\left\|\overrightarrow{\gamma} \wedge \overrightarrow{V}\right\|}{\left\|\overrightarrow{V}\right\|^3} \tag{4}$$

where $\Re$ represents the *radius of curvature*.

A *trajectory curve* whose local *curvature* is null behaves locally as a straight line.

C.3 Torsion of the trajectory curve

Let's consider the *trajectory curve* $\vec{\chi}(t)$ having in M an instantaneous velocity vector $\overrightarrow{V}(t)$, an instantaneous acceleration vector $\overrightarrow{\gamma}(t)$, and an instantaneous over-acceleration vector $\dot{\vec{\gamma}}$, the *torsion*, which expresses the difference between the *trajectory curve* and a *plane* curve, is defined by:

$$\frac{1}{\Im} = -\frac{\dot{\vec{\gamma}} \cdot \left(\overrightarrow{\gamma} \wedge \overrightarrow{V}\right)}{\left\|\overrightarrow{\gamma} \wedge \overrightarrow{V}\right\|^2} \tag{5}$$

where $\Im$ represents the *radius of torsion*.

A *trajectory curve* whose local *torsion* is null behaves locally as a plane curve.

D. Slow Manifold Analytical Equation

Various methods of determination of the *slow manifold* analytical equation of systems (1) have been developed during the second part of the last century. *Differential Geometry method* which has been subject to a publication [4] provides a direct determination of the *slow manifold* analytical equation for systems (1).

D.1 Slow manifold equation of a two-dimensional singular perturbed system

Proposition II.1: The location of the points where the local curvature of the trajectory curves integral of a two-dimensional singularly perturbed system (1) is null, provides the analytical equation of the slow manifold associated with this system.

The vanishing condition of the *curvature* provides:

$$\frac{1}{\Re} = 0 \ \Leftrightarrow\ \left\|\overrightarrow{\gamma} \wedge \overrightarrow{V}\right\| = 0 \tag{6}$$

D.2 Slow manifold equation of a three-dimensional singular perturbed system

Proposition II.2: The location of the points where the local torsion of the trajectory curves integral of a three dimensional singularly perturbed system (1) is null, provides the analytical equation of the slow manifold associated with this system.

The vanishing condition of the *torsion* provides:

$$\frac{1}{\Im} = 0 \Leftrightarrow \dot{\vec{\gamma}} \cdot \left(\vec{\gamma} \wedge \vec{V} \right) = 0 \tag{7}$$

Proof. For proofs of these two propositions see Ginoux *et al.* [4].

Let's notice that Proposition II.1 and II.2 provide *slow manifold implicit equation* which reads:

$$\phi \left(\vec{X} \right) = 0 \tag{8}$$

In his first report entitled on the "Courbes définies par une équation différentielle" Henri Poincaré [10] proposed a criterion making it possible to characterize the attractivity or the repulsivity of a manifold.

D.3 Attractive, repulsive manifolds

Proposition II.3: Let $\vec{X}(t)$ be a trajectory curve having in M an instantaneous velocity vector $\vec{V}(t)$ and let $(\mathcal{V})$ be a manifold (a curve in dimension two, a surface in dimension three) defined by the implicit equation $\phi = 0$ whose normal vector $\vec{\eta} = \vec{\nabla}\phi$ is directed towards the outside of the concavity of this manifold.

- *If the scalar product between the instantaneous velocity vector $\vec{V}(t)$ and the normal vector $\vec{\eta} = \vec{\nabla}\phi$ is positive, the manifold is said attractive with respect to this trajectory curve*

- *If it is null, the trajectory curve is tangent to this manifold.*

- *If it is negative, the manifold is said repulsive. This scalar product which represents the total derivative of ϕ constitutes a new manifold $(\dot{\mathcal{V}})$ which is the envelope of the manifold $(\mathcal{V})$.*

Proposition II.3 makes it is possible to specify the attractive (resp. repulsive) feature of the *slow manifold analytical equation* defined by Proposition II.1 or II.2, i.e., enables to discriminate its stable and unstable parts.

Moreover, according to Fenichel's theorem for the persistence of normally hyperbolic invariant manifolds [3], it has been established in [4] on the one hand that the *slow manifold* equation is invariant and, on the other hand according to Proposition II.3, that the attractive part of this *slow invariant manifold* is the exact *slow manifold* equation associated with systems (1).

III. APPLICATIONS

Differential Geometry method is thus applied to nonlinear electronics systems in order to determine their *slow invariant manifold* analytical equation and its stable and unstable parts.

A. Van der Pol model

The oscillator of B. Van der Pol [16] is a second-order system with non-linear frictions which can be written:

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

The particular form of the friction which can be carried out by an electric circuit causes a decrease of the amplitude of the great oscillations and an increase of the small. There are various manners of writing the previous equation like a first order system. One of them is:

$$\begin{cases} \dot{x} = \alpha \left(x + y - \frac{x^3}{3} \right) \\ \dot{y} = -\frac{x}{\alpha} \end{cases}$$

When α becomes very large, x becomes a "fast" variable and y a "slow" variable. In order to analyze the limit $\alpha \to \infty$, we introduce a small parameter $\varepsilon = 1/\alpha^2$ and a "slow time" $t' = t/\alpha = \sqrt{\varepsilon} t$. Thus, the system can be written:

$$\vec{V} \begin{pmatrix} \varepsilon \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} x + y - \frac{x^3}{3} \\ -x \end{pmatrix} \tag{9}$$

with ε a positive real parameter ($\varepsilon = 0.05$) where the functions f and g are infinitely differentiable with respect to all x_i and t, i.e., are C^∞ functions in a compact E included in $\mathbb{R}^{\not{k}}$ and with values in $\mathbb{R}$. Applying *Differential Geometry method*, Proposition II.1 leads to the following *slow invariant manifold* analytical equation:

$$9y^2 + \left(9x + 3x^3\right) y + 6x^4 - 2x^6 + 9x^2\varepsilon = 0 \tag{10}$$

Since this equation is quadratic in y, it can be solved in order to plot y according to x.

$$y_{1,2} = -\frac{x^3}{6} - \frac{x}{2} \pm \frac{x}{2}\sqrt{x^4 - 2x^2 + 1 - 4\varepsilon} \quad (11)$$

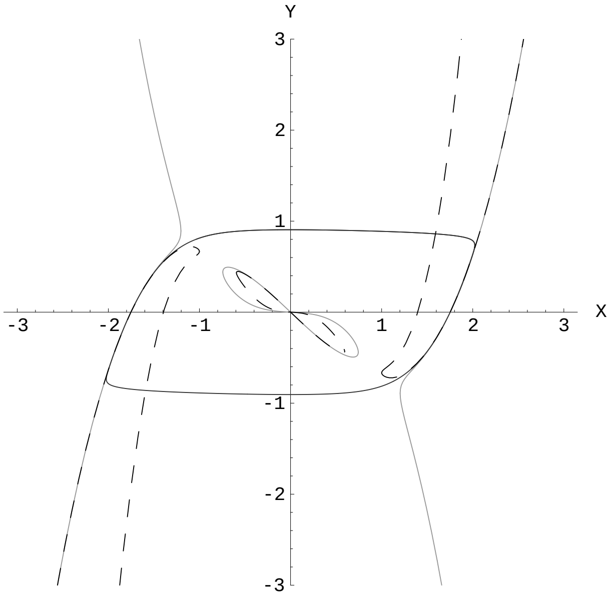

Fig. 1. Slow manifold of the Van der Pol system (9).

In Fig. 1 is plotted the *slow manifold* equation (11) of the Van der Pol system with $\varepsilon = 0.05$ by using Proposition II.1, i.e., the collinearity condition between the instantaneous velocity vector $\overrightarrow{V}$ and the instantaneous acceleration vector $\overrightarrow{\gamma}$, i.e., the location of the points where the *curvature* of the *trajectory curves* is null. Moreover, Proposition 3 enables to specify its attractive (dashed line) and repulsive parts which are corresponding to its stable and unstable parts.

B. Colpitts oscillator

The Colpitts oscillator considered in this example contains a bipolar junction transistor (BJT) as the gain element and a resonant nework consisting of an inductor and a pair of capacitors. Illustrations and details may be found in [8]. Note that the bias is provided by the current source I_0, characterized by a Norton-equivalent conductance G_0. Let's make the following assumptions.

- (H_1) The (BJT) is modeled by a voltage-controlled nonlinear resistor and a linear current-controlled current source, that is, the parasitic and reversed effects are discarded.

- (H_2) Ideal bias circuit, i.e., the bias current on the emitter is provided by an ideal current source, I_0 with $G_0 = 0$.

- (H_3) The common-base forward short-circuit current again is ideal.

The state equations of this model are the following:

$$\left\{ \begin{array}{l} C_1 \frac{dV_{C_1}}{dt} = -f\left(V_{C_2}\right) + I_L \\ C_2 \frac{dV_{C_2}}{dt} = I_L - I_0 \\ L\frac{dI_L}{dt} = -V_{C_1} - V_{C_2} - RI_L + V_{CC} \end{array} \right. \quad (12)$$

Where $f(\cdot)$ is the driving-point characterizing of the nonlinear resistor and may be expressed in the form: $I_E = f\left(V_{C_2}\right)$

Introducing a set of dimensionless state variables (x, y, z) and choosing the operating point of (12) to be the origin of the new coordinate system, the state equations of the Colpitts oscillator can be rewritten in the form:

$$\left\{ \begin{array}{l} \dot{x} = \frac{g^*}{Q(1-k)}\left(-n\left(y\right) + z\right) \\ \dot{y} = \frac{g^*}{Qk} z \\ \dot{z} = -\frac{Qk(1-k)}{g^*}\left(x + y\right) - \frac{1}{Q} z \end{array} \right. \quad (13)$$

where $n(y) = \exp(-y) - 1$, $k = C_1/(C_1 + C_2)$, g^* is the loop gain of the oscillator when the phase condition of the Barkhausen criterion is satisfied,

$Q = \omega_0 L/R$ the quality factor of the unloaded tank circuit ($\omega_0 = 1\Big/\sqrt{LC_1C_2/(C_1+C_2)}$).

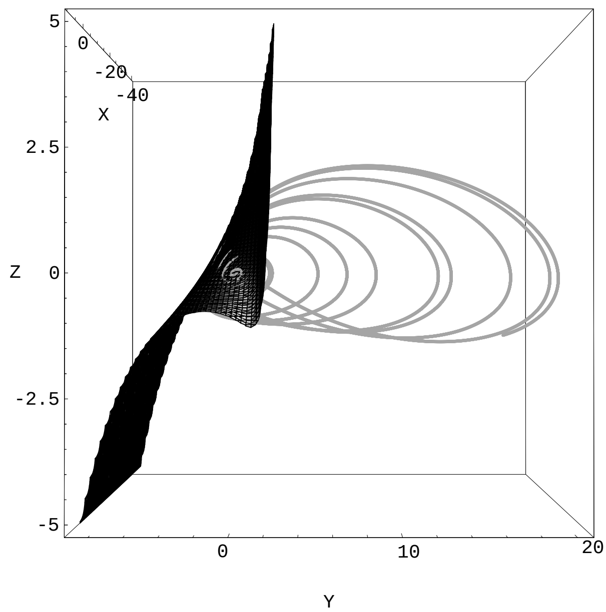

Fig. 2. Slow manifold of the Colpitts oscillator (13).

In Fig. 3 is plotted with the following set of parameters: $log_{10}g^* = 0.5$, $log_{10}Q = 0.1510$ and $k = 1/2$, the *slow manifold* equation of the Colpitts oscillator by using Proposition II.2. This cumbersome equation which has not been expressed constitutes the coplanarity condition between the instantaneous velocity vector $\overrightarrow{V}$, the instantaneous acceleration vector $\overrightarrow{\gamma}$ and the instantaneous over-acceleration vector $\dot{\vec{\gamma}}$, i.e., the location of the points where the *torsion* of the *trajectory curves* is null. Moreover, Proposition II.3 enables to specify its attractive and repulsive parts which are corresponding to its stable and unstable parts.

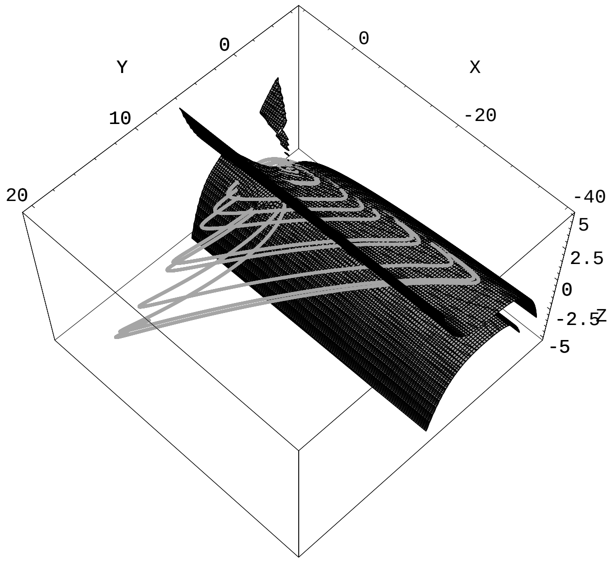

Fig. 3. Stable part of the slow manifold of the Colpitts oscillator (13).

IV. DISCUSSION

In this work a new approach based on the use of local metrics properties of curvature and torsion resulting from Differential Geometry has been applied to electronics systems. Considering trajectory curves as plane or space curves, these properties have directly provided the *slow invariant manifold* analytical equation associated with such systems and its stable and unstable parts. Van der Pol relaxation oscillator which has been extensively studied, plays the role of a paradigm for singularly perturbed system, and has been presented in order to emphasize this approach for two-dimensional systems. On the contrary, the Colpitts (harmonic) oscillator can not be considered as a singularly perturbed system, but *Differential Geometry method* enabled to provide the first determination of its *slow invariant manifold* analytical equation and its stable and unstable parts.

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