SINGULAR MANIFOLDS AND ATTRACTORS STRUCTURE IN PREDATOR-PREY MODELS

Jean-Marc Ginoux
PROTEE Laboratory, I.U.T., Université du Sud,
B.P. 20132, 83957, LA GARDE Cedex, France
ginoux@univ-tln.fr

Bruno Rossetto
PROTEE Laboratory, I.U.T., Université du Sud,
B.P. 20132, 83957, LA GARDE Cedex, France
rossetto@univ-tln.fr

Abstract – We consider three - trophic food chains, particularly Rosenzweig - MacArthur, Hastings - Powell and Volterra-Gause models, interacting in a predator prey mode. Kinematics methods, especially acceleration properties, carry out a determination of the slow manifold in a simple way and provide a criterion of its attractivity. While using the Frénet frame, we give a qualitative description of attractors as the deployment in the phase space of a submanifold, called singular manifold.

I. Dynamical systems

In the following we consider a system of differential equations defined in a compact E included in $\mathbb{R}^n$:

$$\frac{d\vec{X}(t)}{dt} = \vec{\Im}(\vec{X}) \quad (1)$$

with

$$\vec{X} = {}^t\left[x_1, x_2, ..., x_n\right] \in E \subset \mathbb{R}^n$$

and

$$\vec{\Im}(\vec{X}) = {}^t\left[\, f_1(\vec{X}), f_2(\vec{X}), ..., f_n(\vec{X})\right] \in E \subset \mathbb{R}^n$$

The vector $\vec{\Im}$ defines a velocity vector field in E whose components f_i which are supposed to be continuous and infinitely derivable with respect to all x_i, i.e., are $\mathbf{C}^\infty$ functions in E, with values included in $\mathbb{R}$, and checking the assumptions of the Cauchy-Lipschitz theorem. For more details, see for example [1].

A solution of this system is an integral curve $\vec{X}(t)$ tangent to $\vec{\Im}$ whose values define the *states* of the *dynamical system* described by the Eq. (1). Since none of the components f_i of the velocity vector field depends here explicitly on time, the system is *autonomous*.

II. Kinematics vector functions

The approach suggested consists in using the *Mechanics* formalism. So, it is necessary to define the kinematics variables needed to its development. Thus, we can assimilate the solution of a dynamical system to the co-ordinates, i.e., to the position, of a point M at the instant *t*. This integral curve defined by $\vec{X}(t)$ the vector function of the scalar variable *t* represents the *trajectory* of M.

A. Instantaneous velocity vector

As the vector function $\vec{X}(t)$ of the scalar variable *t* represents the *trajectory* of M, the total differential of $\vec{X}(t)$ is the vector function $\vec{V}(t)$ of the scalar variable *t* which represents the instantaneous velocity vector of the mobile M at the instant *t*.
We will note:

$$\vec{V}(t) = \frac{d\vec{X}(t)}{dt} = \vec{\Im}(\vec{X}) \quad (2)$$

The instantaneous velocity vector $\vec{V}(t)$ is supported by the tangent to the *trajectory*.

B. Instantaneous acceleration vector

As the instantaneous vector function $\vec{V}(t)$ of the scalar variable *t* represents the velocity vector of M, the total differential of $\vec{V}(t)$ is the vector function $\vec{\gamma}(t)$ of the scalar variable *t* which represents the instantaneous acceleration vector of the mobile M at the instant *t*. We will note:

$$\vec{\gamma}(t) = \frac{d\vec{V}(t)}{dt} \quad (3)$$

Since the functions f_i are supposed to be $\mathbf{C}^\infty$ functions in a compact E included in $\mathbb{R}^n$, it is possible to calculate the total differential of the vector field $\vec{\Im}$ defined by (2). By using the derivation of the composed functions, a derivative within the meaning of Fréchet appears:

$$\frac{d\vec{V}}{dt} = \frac{d\vec{\Im}}{d\vec{X}} \frac{d\vec{X}}{dt} \quad (4)$$

while noticing that $\frac{d\vec{\Im}}{d\vec{X}}$ is the functional jacobian matrix J of the system (1), and according to Eqs. (2) and (3) we have the following equation which plays a very important role:

$$\vec{\gamma} = J\vec{V} \quad (5)$$

III. Frénet frame

Using the Frénet basis, i.e., a basis built starting from the *trajectory* curve $\vec{X}(t)$ directed towards the motion of the mobile M. Let's define $\vec{\tau}$ the unit vector of the tangent to the *trajectory* curve in M, $\vec{\nu}$ the unit vector of the principal normal in M directed towards the interior of the concavity of the curve and $\vec{\beta}$ the unit vector of the binormal to the *trajectory* curve in M so that the trihedron ($\vec{\tau}, \vec{\nu}, \vec{\beta}$) is direct. Since the instantaneous velocity vector $\vec{V}(t)$ is tangent for any point M to the *trajectory* curve $\vec{X}(t)$, we can choose a tangent unit vector as following:

$$\vec{\tau} = \frac{\vec{V}}{\|\vec{V}\|} \quad (6)$$

In the same manner, we can choose a unit vector of the normal, as:

$$\vec{\nu} = \frac{{}^{\perp}\vec{V}}{\|{}^{\perp}\vec{V}\|} \quad (7)$$

with

$$\|\vec{V}\| = \|{}^{\perp}\vec{V}\|$$

where the vector ${}^{\perp}\vec{V}$ represents the normal vector to the instantaneous velocity vector $\vec{V}$ directed towards the interior of the concavity of the curve. Thus, we can write the tangential and normal components of the instantaneous acceleration vector
$\vec{\gamma}$ as:

$$\gamma_\tau = \frac{\vec{\gamma}.\vec{V}}{\|\vec{V}\|} \qquad \gamma_\nu = \frac{\|\vec{\gamma} \wedge \vec{V}\|}{\|\vec{V}\|} \quad (7)$$

By noticing that the variation of the norm of the instantaneous velocity vector can be written:

$$\frac{d\|\vec{V}\|}{dt} = \frac{\vec{\gamma}.\vec{V}}{\|\vec{V}\|} = \gamma_\tau$$

Moreover, in the *Mechanics* formalism the reciprocal radius of curvature and the torsion may be written:

$$\frac{1}{\Re} = \frac{\|\vec{\gamma} \wedge \vec{V}\|}{\|\vec{V}\|^3} = \frac{\gamma_\nu}{\|\vec{V}\|^2} \qquad \Im = -\frac{\dot{\vec{\gamma}}.\left(\vec{\gamma} \wedge \vec{V}\right)}{\|\vec{\gamma} \wedge \vec{V}\|^2} \quad (8)$$

IV. New Method of Determination of the Slow Manifold Equation using Acceleration properties

In the *Mechanics* formalism, the study of the motion of a mobile M consists in being interested in the variation of the Euclidian norm of its instantaneous velocity vector $\vec{V}$, i.e., in the tangential γ_τ component of its instantaneous acceleration vector $\vec{\gamma}$. The variation of the Euclidian norm of the instantaneous velocity vector $\vec{V}$ depends on the sign of the scalar product between the instantaneous velocity vector $\vec{V}$ and the instantaneous acceleration vector $\vec{\gamma}$, i.e., of the angle formed by these two vectors. Thus if, $\vec{\gamma}.\vec{V} > 0$, the variation of the Euclidian norm of the instantaneous velocity vector $\vec{V}$ is positive and the Euclidian norm of the instantaneous velocity vector $\vec{V}$ increases. The motion is accelerated, it is in its *fast* phase. If, $\vec{\gamma}.\vec{V} = 0$ the variation of the Euclidian norm of the instantaneous velocity vector $\vec{V}$ is null and the Euclidian norm of the instantaneous velocity vector $\vec{V}$ is constant. The motion is *uniform*, it is in a phase of transition between its *fast* phase and its *slow* phase. Moreover, the instantaneous velocity vector $\vec{V}$ is perpendicular to the instantaneous acceleration vector $\vec{\gamma}$. If, $\vec{\gamma}.\vec{V} < 0$, the variation of the Euclidian norm of the instantaneous velocity vector $\vec{V}$ is negative and the Euclidian norm of the instantaneous velocity vector $\vec{V}$ decreases. The motion is decelerated, it is in its slow phase. So, the study of the nature of the motion of a mobile M depends on the sign of the scalar product between the instantaneous velocity vector $\vec{V}$ and the instantaneous acceleration vector $\vec{\gamma}$, i.e., of the angle formed by these two vectors. Still using the *Mechanics* formalism, we will now focus our attention on the directions of these two vectors and use the fact that this scalar product is maximum when these vectors are collinear. So, if the instantaneous velocity vector $\vec{V}$ and the instantaneous acceleration vector $\vec{\gamma}$ are collinear and of opposite directions the motion is decelerated. These features make it possible to delimit the *slow* and *fast* domains of the phase space.

Proposition 1.

If $I(x_0, y_0, z_0)$ is one of the equilibrium points of a dynamical system defined by Eq. (1) and represented by the instantaneous velocity vector $\vec{V}$ from which the instantaneous acceleration vector $\vec{\gamma}$ is deduced, then, the plane (P) going through the fixed point $I(x_0, y_0, z_0)$ and having for direction vectors the instantaneous vectors $\vec{V}$ and $\vec{\gamma}$ is defined by the coplanarity condition between $\vec{V}$, $\vec{\gamma}$ and $\overrightarrow{IM}$ formed starting from any fixed point I and from any point $M(x, y, z)$ belonging to (P). So,

$$\forall\, M \in (P) \;\Leftrightarrow\; \exists\, (\mu, \eta) \in \mathbb{R}^2 \,/\, \overrightarrow{IM} = \mu\vec{V} + \eta\vec{\gamma}$$

This coplanarity condition may be written:

$$\overrightarrow{IM} \cdot \left(\vec{V} \wedge \vec{\gamma}\right) = 0 \qquad (9)$$

Thus, this equation defines an attractive slow manifold in the case of slow-fast autonomous dynamical system (S-FADS), i.e., systems having a small parameter in one of the component of velocity or systems which can be considered as (S-FADS) [8]. The proof of this proposition and following are stated in other papers.

V. Singular Approximation of the acceleration

It has been shown [8] that for a model having a small parameter ε in one of the components its instantaneous velocity vector $\vec{V}$ the slow manifold equation associated to the singular approximation of the intantaneous velocity vector constituted the zero-order approximation in ε of the slow manifold equation. We will now focus on the slow manifold equation associated to the singular approximation of the instantaneous acceleration vector $\vec{\gamma}$.

Proposition 2.

If a dynamical system has a small parameter ε in one of the components its instantaneous velocity vector $\vec{V}$ and so, a slow manifold equation associated to the singular approximation, the slow manifold equation associated to the singular approximation of the instantaneous acceleration $\vec{\gamma}$ constitutes the first-order approximation in ε of the slow manifold. Thus, the slow manifold can be obtained easier while writing:

$$\frac{d^2x}{dt^2} = 0 \qquad (10)$$

It has been previously stated that the equation defined by proposition 2 leads to the first-order approximation in ε of the equation of the manifold (V_1) which we called first-order approximation of the equation of the slow manifold (V_0). It is possible to show that the successive approximations of the equations of the manifolds (V_2), (V_3), etc are obtained respectively from $\dot{\gamma} = 0$, $\ddot{\gamma} = 0$, etc. and are the envelopes of the slow manifold (V_0).

The manifold (V_1) obtained by proposition 2 is locally tangent to the slow manifold (V_0). These manifolds are thus locally parallel. The theory of the parallel curves and surfaces (Leibniz) show that the distance between these manifolds is cancelled when the assumptions of proposition 1 are checked. Consequently what we call the first-order approximation (V_1) coincides locally with the slow manifold (V_0).

VI. Singular manifold

The singular manifold is defined as the location of the points belonging to the slow manifold of equation $\Phi = 0$ and for which the tangential component γ_τ of the instantaneous acceleration vector $\vec{\gamma}$ is cancelled. This one-dimensional manifold is a submanifold of the slow manifold. Let us consider the location of the points obtained by integration in a given time of the system from initial conditions taken on this manifold. They constitute a submanifold which also belongs to the attractor. The whole of these manifolds corresponds to different points of integration making it possible to reconstitute the attractor by redeployment of the singular manifold.

VII. Application to predator-prey models

A. Rosenzweig-MacArthur model

This model, elaborated by M. L. Rosenzweig and R.H. MacArthur [7], starting from the equations of V. Volterra [10] transcribes the evolution of three species: x, y and z interacting in a predator-prey mode and is composed of a Verhulst [9] logistic prey x, a predator y, and top-predatory z, whose predation are limited by functional responses of Holling [5] *type 2*.

$$\vec{V}\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} \frac{x}{\xi}\left(1 - x - \frac{y}{x+\beta_1}\right) \\ y\left(\frac{x}{x+\beta_1} - \delta_1 - \frac{z}{y+\beta_2}\right) \\ \varepsilon z\left(\frac{y}{y+\beta_2} - \delta_2\right) \end{pmatrix} \qquad (10)$$

with

$$\xi = 0.1,\ \varepsilon = 0.3,\ \beta_1 = 0.3,\ \beta_2 = 0.1,\ \delta_1 = 0.1,\ \delta_2 = 0.62$$

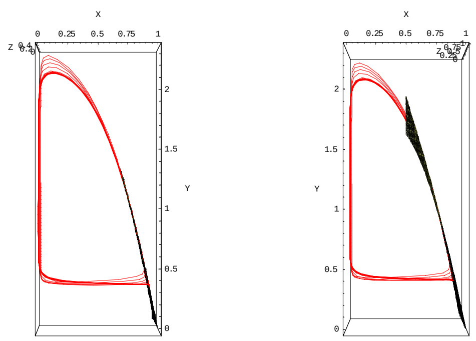

Fig. 1: Slow manifold and Singular approximation of the instantaneous acceleration vector of the Rosenzweig-MacArthur model

B. Hastings-Powell model

By carrying out some change of variables in the Rosenzweig-Mac Arthur [7] model, one obtains the Hastings and Powell [4] model

$$\vec{V}\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{pmatrix} = \begin{pmatrix} x\left(1-x-\frac{a_1 y}{1+x\beta_1}\right)\\ y\left(\frac{a_1 x}{1+x\beta_1}-\delta_1-\frac{a_2 z}{1+y\beta_2}\right)\\ z\left(\frac{a_2 y}{1+y\beta_2}-\delta_2\right)\end{pmatrix} \qquad (11)$$

with

$a_1 = 5,\ a_2 = 0.1,\ \beta_1 = 3,\ \beta_2 = 2,\ \delta_1 = 0.4,\ \delta_2 = 0.01$

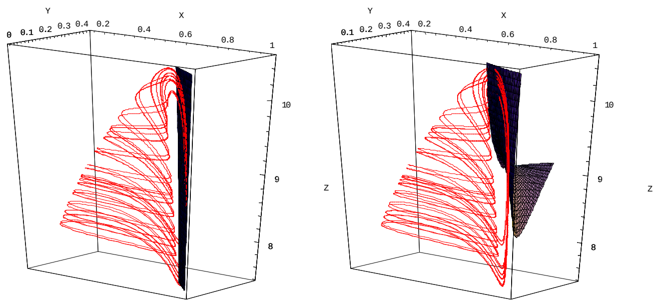

Fig. 2: Slow manifold and singular approximation of the instantaneous acceleration vector of the Hastings-Powell model

C. Volterra-Gause model

This model elaborated by J.-M. Ginoux, B. Rossetto and J.-L. Jamet [3] which consists of a prey, a predator and top-predator, has been named Volterra-Gause since it is made up of the original model of V. Volterra [10] incorporating a logisitic limitation of P.F. Verhulst [9] type on the growth of the prey and a limitation of G.F. Gause [2] type on the intensity of the predation of the predator on the prey and of top-predator on the predator.

$$\vec{V}\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\\ \frac{dz}{dt}\end{pmatrix} = \begin{pmatrix} \frac{1}{\xi}\left(x\left(1-x\right)-x^{1/2}y\right)\\ -y\delta_1+x^{1/2}y-y^{1/2}z\\ \varepsilon z\left(y^{1/2}-\delta_2\right)\end{pmatrix} \qquad (12)$$

with $\xi = 0.866,\ \varepsilon = 1.428,\ \delta_1 = 0.577,\ \delta_2 = 0.376$

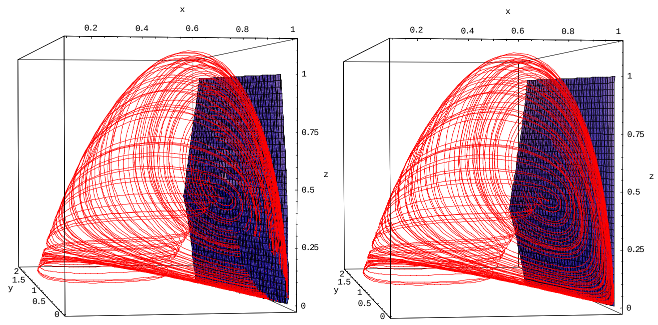

Fig. 3: Slow manifold and singular approximation of the instantaneous acceleration vector of the Volterra-Gause model

The use of the algorithm developed by [11] made it possible to compute what can be regarded as its Lyapunov exponents: (+0.035, 0.000, -0.628). Then, the Kaplan-Yorke [6] conjecture provided the following Lyapunov dimension: 2.06. So, the fractal dimension of this chaotic attractor is close to that of a surface and it is thus possible to consider a deployment of a singular manifold. Now, let us consider the location of the points obtained by integration in a given time of initial conditions taken on this manifold. They constitute a submanifold which also belongs to the attractor. The whole of these manifolds corresponds to different points of integration making it possible to reconstitute the attractor by redeployment of the singular manifold. In order to illustrate this concept let's apply it on this predator-prey model.

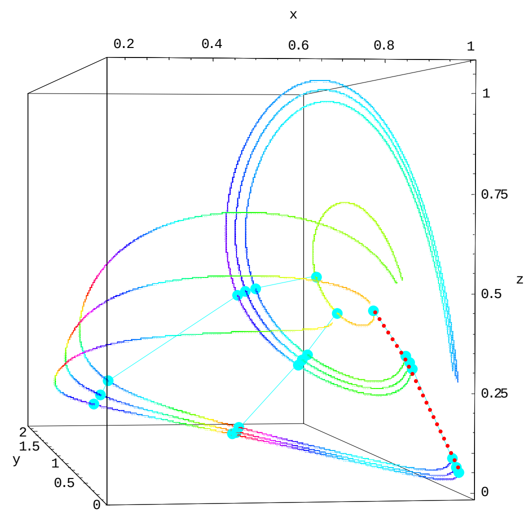

Fig. 4: Deployment of the singular manifold (dot line) of the Volterra-Gause model

VIII. Discussion

In this work the use of the *Mechanics* formalism and of the instantaneous acceleration vector provided on the one hand a condition of discrimination of the *slow* and *fast* domains of the phase space, and on the other hand, a new condition of determination of the slow manifold equation of (S-FADS) or of dynamical systems considered as (S-FADS): the coplanarity between the instantaneous velocity vector $\vec{V}$, the instantaneous acceleration vector $\vec{\gamma}$ and the vector

$\overrightarrow{IM}$ formed starting from a fixed point of such dynamical systems and any point M.
Moreover, this *kinematics method* provided for such systems a new manifold: the singular approximation of the instantaneous acceleration vector $\vec{\gamma}$ constituting the first-order approximation in ε of the slow manifold obtained by the coplanarity condition. At last, the introduction of the singular manifold which can reconstitute the attractor by successive integrations of points taken on this manifold, i.e., by redeployment, provides a qualitative description of its structure. The *Mechanics* formalism and more precisely the radius of curvature and the torsion could be useful to go further in the description of the deployment and thus in the understanding of the attractor structure.
This work has highlighted too certain similarities between three different models. Their common features and the possibility of transition between one to the other by a simple variation of parameter offer an alternative for the choice of the model.
This could be very useful for the biologists who work with predator-prey models. In spite of differences in their functional responses these models present some striking similarities in the nature and the number of their fixed points, like in their dynamic behavior: existence of a limit cycle, occurrence of a Hopf bifurcation, presence of a chaotic attractor or period doubling cascades.

Dynamical features \ Models	Rosenzweig – Mac Arthur		Hastings – Powell		Volterra – Gause	
Equilibrium points	$O(0, 0, 0)$	$I(\hat{x}, \hat{y}, 0)$	$O(0, 0, 0)$	$I(\hat{x}, \hat{y}, 0)$	$O(0, 0, 0)$	$I(\hat{x}, \hat{y}, 0)$
	$J(x^*, y^*, z^*)$	$K(1, 0, 0)$	$J(x^*, y^*, z^*)$	$K(1, 0, 0)$	$J(x^*, y^*, z^*)$	$K(1, 0, 0)$
Attractional sink	2		2		2	
Hopf bifurcation	$\delta_1 = 0.6835$		$\delta_1 = 0.7402$		$\delta_1 = 0.7474$	
Chaotic attractor	Moebius strip		Tea cup		Snail shell	
Period – doubling	$\delta_2 = 0.67785$		$b_1 = 2.437$		$\delta_1 = 0.625$	
Slow manifold	1		1		1	

Both fixed points O (0, 0, 0) present the same stability, i.e., they have attractive directions according to z'z and repulsive directions according to x'x. Eigendirections of points K (1, 0, 0) are attractive according to x'x and z'z. Points I and J (x^*, y^*, z^*) have the behaviour of a stable and an unstable focus (resp.), one in the *xy* plane and the other apart from the *xy* plane. These models introduce rich and complexes dynamics of which it remains still much of aspects to study.

Moreover it appears that it is possible, in some domains of variation of parameters, to reduce the models dimension which will be useful to take in account the influence of the external medium by time-dependent coefficients.

IX. References

[1] E.A. Coddington & N. Levinson, *Theory of Ordinary Differential Equations*, Mac Graw Hill, New York, 1955.

[2] G.F. Gause, *The struggle for existence*, Williams and Wilkins, Baltimore, 1935.

[3] J.M. Ginoux, B. Rosetto &J.L. Jamet, "Chaos in a three-dimensional Volterra-Gause model of predator-prey type, " *Int. J. Bifurcations and Chaos* **5** , Vol.15, "in press", 2005.

[4] A. Hastings & T. Powell, "Chaos in a three-species food chain," *Ecology* **72** , 896-903, 1991.

[5] C.S. Holling, "Some characteristics of simple types of predation and parasitism," *Can. Entomologist* **91**, 385-398, 1959.

[6] J. Kaplan & J.A. Yorke, "Chaotic behavior of multidimensional difference equations, in functional differential equations and approximation of fixed points," *Lecture Notes in Mathematics* **730**, 204-227, 1979.

[7] M.L. Rosenzweig & R.H. Mac Arthur, "Graphical representation and stability conditions of predator-prey interactions," *Am. Nat.* **97** , 209-223, 1963.

[8] S. Ramdani, B. Rossetto, L.O. Chua & R. Lozi, "Slow Manifold of Some Chaotic Systems-Laser Systems Applications," *Int. J. Bifurcations and Chaos* **10** , Vol.12, 2729-2744, 2000.

[9] P.F. Verhulst, "Notice sur la loi que suit la population dans son accroissement," *Corresp. Math. Phys.* **X**, 113-121 [in French], 1838.

[10] V. Volterra, "Variazioni e fluttuazioni del numero d'individui in specie animali conviventi," *Mem. Acad. Lincei* III **6** , 31-113 [in Italian], 1926.

[11] A. Wolf, J.B. Swift, H.L. Swinney & J.A. Vastano, "Determining Lyapunov Exponents from a Time Series," *Physica D*, Vol. 16, 285-317, 1985.